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THE DECOMPOSITION THEOREM, PERVERSE SHEAVES AND THE TOPOLOGY OF ALGEBRAIC MAPS

MARK ANDREA A. DE CATALDO AND LUCA MIGLIORINI

Abstract. We give a motivated introduction to the theory of perverse sheaves, culminating in the decomposition theorem of Beilinson, Bernstein, Deligne and Gabber. A goal of this survey is to show how the theory develops naturally from classical constructions used in the study of topological properties of algebraic varieties. While most proofs are omitted, we discuss several approaches to the decomposition theorem and indicate some important applications and examples.

Contents

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1. Overview

The theory of perverse sheaves and one of its crowning achievements, the decomposition theorem, are at the heart of a revolution which has taken place over the last thirty years in algebra, representation theory and algebraic geometry.

The decomposition theorem is a powerful tool for investigating the topological properties of proper maps between algebraic varieties and is the deepest known fact relating their homological, Hodge-theoretic and arithmetic properties.

In this §1, we try to motivate the statement of this theorem as a natural outgrowth of the investigations on the topological properties of algebraic varieties begun with Lefschetz and culminating in the spectacular results obtained with the development of Hodge theory and étale cohomology. We gloss over many crucial technical details in favor of rendering a more panoramic picture; the appendices in §5 offer a partial remedy to these omissions. We state the classical Lefschetz and Hodge theorems for projective manifolds in §1.1 and Deligne's results on families of projective manifolds in §1.2. In §1.3, we briefly discuss singular varieties and the appearance and role of mixed Hodge structures and intersection cohomology. In §1.4, we state the decomposition theorem in terms of intersection cohomology without any reference to perverse sheaves. The known proofs, however, use in an essential way the theory of perverse sheaves, which, in turn, is deeply rooted in the formalism of sheaves and derived categories. We offer a "crash course" on sheaves in §1.5. With these notions and ideas in hand, in §1.6 we state the decomposition theorem in terms of intersection complexes (rather than in terms of intersection cohomology groups). We also state two important related results: the relative hard Lefschetz and semisimplicity theorems. §1.7 reviews the generalization to singular maps of the now classical properties of the monodromy representation in cohomology for a family of projective manifolds. §1.8 discusses surface and threefold examples of the statement of the decomposition theorem. §1.9 overviews the mixed Hodge structures arising from the decomposition theorem. We provide a timeline for the main results mentioned in this overview in §1.10.

We have tried, and have surely failed in some ways, to write this survey so that most of it can be read by nonexperts and so that each chapter can be read independently of the others. For example, a reader interested in the decomposition theorem and in its applications could read §1, the first half of §4 and skim through the second half on geometrization, while a reader interested in the proofs could read §1 and §3. Perhaps, at that point, the reader may be motivated to read more about perverse sheaves.

§2 is an introduction to perverse sheaves. In this survey, we deal only with middle perversity, i.e. with a special case of perverse sheaves. It seemed natural to us to start this section with a discussion of intersection cohomology. In §2.3, we define perverse sheaves and discuss their first properties, as well as their natural categorical framework, i.e. t-structures. In §2.4, we introduce the perverse filtration in cohomology and its geometric description via the Lefschetz hyperplane theorem. §2.5 reviews the basic properties of the cohomology functors associated with the perverse t-structure. §2.6 is about the Lefschetz hyperplane theorem for intersection cohomology. In §2.7, we review the properties of the intermediate extension functor, of which intersection complexes are a key example.

In §3, we discuss the three known approaches to the decomposition theorem: the original one, due to A. Beilinson, J. Bernstein, P. Deligne and O. Gabber, via the arithmetic properties of varieties over finite fields, the one of M. Saito, via mixed Hodge modules, and ours, via classical Hodge theory. Each approach highlights different aspects of this important theorem.

§4 contains a sampling of applications of the theory of perverse sheaves and, in particular, of the decomposition theorem. The applications range from algebraic geometry to representation theory and to combinatorics. While the first half of §4, on toric and on semismall maps, is targeted to a general audience, the second half, on the geometrization of Hecke algebras and of the Satake isomorphism, is technically more demanding. Due to the fact that the recent and exciting development [152] in the Langlands program makes use of a result that deals with the decomposition theorem with "large fibers," we have included a brief discussion of B.C. Ngô's support theorem in §4.6.

The appendix §5 contains a brief definition of quasi-projective varieties (§5.1), of pure and mixed Hodge structures, the statement of the hard Lefschetz theorem and of the Hodge-Riemann relations (§5.2), a description of the formalism of derived categories (§5.3), a discussion of how the more classical objects in algebraic topology relate to this formalism (§5.4), a discussion of the nearby and vanishing cycle functors (§5.5), as well as their unipotent counterparts (§5.6), two descriptions of the category of perverse sheaves (§5.7) and, finally, a formulary for the derived category (§5.8).

Unless otherwise stated, a variety is an irreducible complex algebraic variety and a map is a map of varieties. We work with sheaves of rational vector spaces, so that the cohomology groups are rational vector spaces.

1.1. **The topology of complex projective manifolds: Lefschetz and Hodge theorems.** Complex algebraic varieties provided an important motivation for the development of algebraic topology from its earliest days. On the other hand, algebraic varieties and algebraic maps enjoy many truly remarkable topological properties that are not shared by other classes of spaces and maps. These special features were first exploited by Lefschetz ([124]) (who claimed to have "planted the harpoon of algebraic topology into the body of the whale of algebraic geometry" [125], p.13), and they are almost completely summed up in the statement of the decomposition theorem and of its embellishments.

The classical precursors to the decomposition theorem include the theorems of Lefschetz, Hodge, Deligne, and the invariant cycle theorems. In the next few paragraphs, we discuss the Lefschetz and Hodge theorems and the Hodge-Riemann relations. Together with Deligne's Theorem 1.2.1, these precursors are in fact essential tools in the three known proofs (§3) of the decomposition theorem.

Let X be a nonsingular complex n-dimensional projective variety embedded in some projective space $X \subseteq \mathbb{P}^N$, and let $D = \mathcal{H} \cap X$ be the intersection of X with a generic hyperplane $\mathcal{H} \subseteq \mathbb{P}^N$. Recall that we use cohomology with rational coefficients. A standard textbook reference for what follows is [92]; see also [175, 44].

The *Lefschetz hyperplane theorem* states that the restriction map $H^i(X) \to H^i(D)$ is an isomorphism for $i < n-1$ and is injective for $i = n-1$.

The cup product with the first Chern class of the hyperplane bundle gives a mapping $\bigcup c_1(\mathcal{H}) : H^i(X) \to H^{i+2}(X)$ which can be identified with the composition $H^i(X) \to H^i(D) \to H^{i+2}(X)$, the latter being a "Gysin" homomorphism.

The *hard Lefschetz theorem* states that for all $0 \leq i \leq n$ the i-fold iteration of the cup product operation is an isomorphism

$$(\bigcup c_1(\mathcal{H}))^i : H^{n-i}(X) \xrightarrow{\simeq} H^{n+i}(X).$$

The *Hodge decomposition* is a canonical decomposition

$$H^i(X, \mathbb{C}) = \bigoplus_{p+q=i} H^{p,q}(X).$$

The summand $H^{p,q}(X)$ consists of cohomology classes on X which can be represented by a closed differential form on X of type (p,q) (i.e., one whose local expression involves p dz's and q $d\overline{z}$'s).

For every fixed index $0 \leq i \leq n$, define a bilinear form $S^{\mathcal{H}}$ on $H^{n-i}(X)$ by

$$(a,b) \longmapsto S^{\mathcal{H}}(a,b) := \int_X (c_1(\mathcal{H}))^i \wedge a \wedge b = \deg\left([X] \cap ((c_1(\mathcal{H}))^i \cup a \cup b)\right),$$

where $[X]$ denotes the fundamental homology class of the naturally oriented X.

The hard Lefschetz theorem is equivalent to the nondegeneracy of the forms $S^{\mathcal{H}}$.

The *Hodge-Riemann bilinear relations* (§5.2, (38)) establish their signature properties.

1.2. **Families of smooth projective varieties.** If $f : X \to Y$ is a C^∞ fiber bundle with smooth compact fiber F, let $\underline{H}^j(F)$ denote the local system on Y

whose fiber at the point $y \in Y$ is $H^j(f^{-1}(y))$. We have the associated *Leray spectral sequence*

$$E_2^{i,j} = H^i(Y; \underline{H}^j(F)) \Longrightarrow H^{i+j}(X) \tag{1}$$

and the *monodromy representation*

$$\rho_i : \pi_1(Y, y_0) \to GL(H^i(F)). \tag{2}$$

Even if Y is simply connected, the Leray spectral sequence can be nontrivial, for example, the Hopf fibration $f : S^3 \to S^2$.

We define a *family of projective manifolds* to be a proper holomorphic submersion $f : X \to Y$ of nonsingular varieties that factors through some product $Y \times \mathbb{P}^N$ and for which the fibers are connected projective manifolds. The nonsingular hypersurfaces of a fixed degree in some projective space give an interesting example. By a classical result of Ehresmann, such a map is also a C^∞ fiber bundle.

The results that follow are due to Deligne [56, 59]. Recall that a representation is said to be *irreducible* if it does not admit a nontrivial invariant subspace, i.e. if it is simple in the category of representations.

Theorem 1.2.1 (Decomposition and semisimplicity for families of projective manifolds). *Suppose $f : X \to Y$ is a family of projective manifolds. Then*

(1) *The Leray spectral sequence* (1) *degenerates at the E_2-page and induces an isomorphism*
$$H^i(X) \cong \bigoplus_{a+b=i} H^a(Y; \underline{H}^b(F)).$$
(2) *The representation* (2) *is semisimple: it is a direct sum of irreducible representations.*

Part (1) gives a rather complete description of the cohomology of X. Part (2) is remarkable because the fundamental group of Y can be infinite.

Remark 1.2.2. Theorem 1.2.1, part (1) is stated using cohomology. Deligne proved a stronger, sheaf-theoretic statement; see Theorem 5.2.2.

Remark 1.2.3. For singular maps, the Leray spectral sequence is very seldom degenerate. If $f : X \to Y$ is a resolution of the singularities of a projective variety Y whose cohomology has a mixed Hodge structure which is not pure, then f^* cannot be injective, and this prohibits degeneration in view of the edge-sequence.

The following is the global invariant cycle theorem. We shall come back to this later in §1.7, where we give some generalizations, and in §1.10, where we give some references.

Theorem 1.2.4. *Suppose $f : X \to Y$ is a family of projective manifolds. Then*
$$H^i(F_{y_0})^{\pi_1(Y, y_0)} = \operatorname{Im}\{H^i(X) \longrightarrow H^i(F_{y_0})\},$$
i.e. the monodromy invariants are precisely the classes obtained by restriction from the total space of the family.

Although the classical Lefschetz-Hodge theorems described in §1.1 and the results described in this section appear to be very different from each other, the decomposition theorem forms a beautiful common generalization which holds also in the presence of singularities.

1.3. **Singular algebraic varieties.** The Lefschetz and Hodge theorems fail if X is singular. There are two somewhat complementary approaches to generalize these statements to singular projective varieties. They involve mixed Hodge theory [59, 60] and intersection cohomology [86, 87] (see also [19]).

In *mixed Hodge theory* the topological invariant studied is the same as that investigated for nonsingular varieties, namely, the cohomology groups of the variety, whereas the structure with which it is endowed changes. See [69] for an elementary and nice introduction. The (p,q)-decomposition of classical Hodge theory is replaced by a more complicated structure: the rational cohomology groups $H^i(X)$ are endowed with an increasing filtration W (the weight filtration) $W_0 \subseteq W_1 \subseteq \ldots \subseteq W_{2i} = H^i(X)$, and the complexifications of the graded pieces W_k/W_{k-1} have a (p,q)-decomposition of weight k, that is, $p+q=k$. Such a structure, called a *mixed Hodge structure*, exists canonically on any algebraic variety and satisfies several fundamental restrictions on the weights, such as:

(1) if X is nonsingular, but possibly noncompact, then the weight filtration on $H^i(X)$ *starts* at W_i, that is, $W_r H^i(X) = 0$ for $r < i$;
(2) if X is compact, but possibly singular, then the weight filtration on $H^i(X)$ *ends* at W_i, that is, $W_r H^i(X) = W_i H^i(X) = H^i(X)$ for $r \geq i$.

Example 1.3.1. Let $X = \mathbb{C}^*$; then $H^1(X) \simeq \mathbb{Q}$ has weight 2 and the classes in $H^1(X)$ are of type $(1,1)$. Let X be a rational irreducible curve with a node (topologically, this is a pinched torus, or also the two-sphere with the north and south poles identified); then $H^1(X) \simeq \mathbb{Q}$ has weight 0 and the classes in $H^1(X)$ are of type $(0,0)$.

In *intersection cohomology theory*, by contrast, it is the topological invariant which is changed, whereas the (p,q)-structure turns out to be the same. The *intersection cohomology groups* $IH^i(X)$ (§2.1) can be described using geometric "cycles" on the possibly singular variety X, and this gives a concrete way to compute simple examples. There is a natural homomorphism $H^i(X) \to IH^i(X)$ which is an isomorphism when X is nonsingular. The groups $IH^i(X)$ are finite dimensional; they satisfy the Mayer-Vietoris theorem and the Künneth formula. These groups are not homotopy invariant but, in compensation, they have the following additional features: they satisfy Poincaré duality, the Lefschetz theorems and, if X is projective, they admit a pure Hodge structure.

Example 1.3.2. Let X be the nodal curve of Example 1.3.1. Then $IH^1(X) = 0$.

Example 1.3.3. Let $E \subseteq \mathbb{P}^N_{\mathbb{C}}$ be a nonsingular projective variety of dimension $n-1$, and let $Y \subseteq \mathbb{C}^{N+1}$ be its affine cone with vertex o. The intersection cohomology groups can be easily computed (see [19] and also Example 2.2.1):

$$IH^i(Y) = 0 \text{ for } i \geq n, \qquad IH^i(Y) \simeq H^i(Y \setminus \{o\}) \text{ for } i < n.$$

There is a twisted version of intersection (co)homology with values in a local system L defined on a Zariski dense nonsingular open subset of the variety X. Intersection cohomology with twisted coefficients is denoted $IH^*(X,L)$, and it appears in the statement of the decomposition theorem.

1.4. **Decomposition and hard Lefschetz in intersection cohomology.** The decomposition theorem is a result about certain complexes of sheaves on varieties. In this section, we state a provisional, yet suggestive, form that involves only intersection cohomology groups.

Theorem 1.4.1 (Decomposition theorem for intersection cohomology groups). *Let $f : X \to Y$ be a proper map of varieties. There exist finitely many triples (Y_a, L_a, d_a) made of locally closed, smooth and irreducible algebraic subvarieties $Y_a \subseteq Y$, semisimple local systems L_a on Y_a and integer numbers d_a, such that for every open set $U \subseteq Y$ there is an isomorphism*

$$IH^r(f^{-1}U) \simeq \bigoplus_a IH^{r-d_a}(U \cap \overline{Y}_a, L_a). \tag{3}$$

The triples (Y_a, L_a, d_a) are essentially unique, independent of U, and they are described in [48, 51]. Setting $U = Y$ we get a formula for $IH^*(X)$ and therefore, if X is nonsingular, a formula for $H^*(X)$. If $f : X \to Y$ is a family of projective manifolds, then (3) coincides with the decomposition in Theorem 1.2.1, part (1). On the opposite side of the spectrum, if $f : X \to Y$ is a resolution of the singularities of Y, i.e. X is nonsingular and f is an isomorphism outside a closed subvariety of Y, then we can deduce that the intersection cohomology groups $IH^*(Y)$ are direct summands of $H^*(X)$.

If X is singular, then there is no analogous direct sum decomposition formula for $H^*(X)$. Intersection cohomology turns out to be precisely the topological invariant apt to deal with singular varieties and maps. The notion of intersection cohomology is needed even when X and Y are nonsingular, but the map f is not a submersion.

Remark 1.4.2 (**The splitting is not canonical**). The decomposition map (3) is not uniquely defined. This is analogous to the elementary fact that a filtration on a vector space can always be given in terms of a direct sum decomposition, but the filtration does not determine in a natural way the summands as subspaces of the given vector space. In the case when X is quasi-projective, one can make distinguished choices which realize the summands as mixed Hodge substructures of a canonical mixed Hodge structure on $IH^*(X)$ (see [54, 45] and §1.9, 5).

If $\mathcal{L}$ is a hyperplane line bundle on a projective variety Y, then the hard Lefschetz theorem for the intersection cohomology groups of Y holds; i.e., for every integer $k \geq 0$, the i-th iterated cup product

$$c_1(\mathcal{L})^i : IH^{\dim Y - i}(Y) \xrightarrow{\simeq} IH^{\dim Y + i}(Y) \tag{4}$$

is an isomorphism. Recall that intersection cohomology is not a ring; however, the cup product with a cohomology class is well defined and intersection cohomology is a module over cohomology.

The analogue of Theorem 5.2.1(3) (hard Lefschetz, Lefschetz decomposition and Hodge-Riemann relations) holds for the intersection cohomology groups $IH^*(Y)$ of a singular projective variety Y.

1.5. **Crash course on sheaves and derived categories.** The statement of Theorem 1.4.1 involves only the notion of intersection cohomology. We do not know of a general method for proving the decomposition (3) without first proving the analogous decomposition, Theorem 1.6.1, at the level of complexes of sheaves.

The language and theory of sheaves and homological algebra, specifically derived categories and perverse sheaves, plays an essential role in all the known proofs of the decomposition theorem, as well as in its numerous applications.

In this section, we collect the few facts about sheaves and derived categories needed in order to understand the statement of the decomposition Theorem 1.6.1.

We amplify and complement this crash course in the appendices in §5 and in section §2 on perverse sheaves. Standard references are [19, 82, 87, 116, 108].

1. Complexes of sheaves. Most of the constructions in homological algebra involve complexes. For example, if Z is a $\mathcal{C}^\infty$ manifold, in order to compute the cohomology of the constant sheaf $\mathbb{R}_Z$, we replace it by the complex of sheaves of differential forms, and then we take the complex of global sections, i.e. the de Rham complex. More generally, to define the cohomology of a sheaf A on a topological space Z, we choose an injective, or flabby, resolution, for instance the one defined by Godement,

$$0 \longrightarrow A \longrightarrow I^0 \xrightarrow{d^0} \cdots \xrightarrow{d^{i-1}} I^i \xrightarrow{d^i} I^{i+1} \longrightarrow \cdots$$

then consider the complex of abelian groups

$$0 \longrightarrow \Gamma(I^0) \xrightarrow{d^0} \cdots \xrightarrow{d^{i-1}} \Gamma(I^i) \xrightarrow{d^i} \Gamma(I^{i+1}) \longrightarrow \cdots$$

and finally take its cohomology. The derived category is a formalism developed in order to work systematically with complexes of sheaves with a notion of morphism which is far more flexible than that of morphism of complexes; for instance, two different resolutions of the same sheaf are isomorphic in the derived category. Let Z be a topological space. We consider sheaves of $\mathbb{Q}$-vector spaces on Z. A bounded *complex of sheaves* K is a diagram

$$\cdots \longrightarrow K^{i-1} \xrightarrow{d^{i-1}} K^i \xrightarrow{d^i} K^{i+1} \longrightarrow \cdots$$

with $K^i = 0$ for $|i| \gg 0$ and satisfying $d^i \circ d^{i-1} = 0$ for every i. The *shifted* complex $K[n]$ is the complex with $K[n]^i = K^{n+i}$ and differentials $d_{K[n]} = (-1)^n d_K$. Complexes of sheaves form an Abelian category and we may form the *cohomology sheaf* $\mathcal{H}^i(K) = \mathrm{Ker}(d^i)/\mathrm{Im}(d^{i-1})$, which is a sheaf whose stalk at a point $x \in Z$ is the cohomology of the complex of stalks at x.

2. Quasi-isomorphisms and resolutions. A morphism $K \to L$ of complexes of sheaves is a *quasi-isomorphism* if it induces isomorphisms $\mathcal{H}^i(K) \cong \mathcal{H}^i(L)$ of cohomology sheaves, i.e. if the induced map at the level of the stalks of the cohomology sheaves is an isomorphism at each point $z \in Z$. An *injective (flabby, fine) resolution* of a complex K is a quasi-isomorphism $K \to I$, where I is a complex with injective (flabby, fine) components. Such a resolution always exists for a bounded below complex. The cohomology groups $H^*(Z, K)$ of K are defined to be the cohomology groups of the complex of global sections $\Gamma(I)$ of I. As soon as one identifies sheaves with the complexes of sheaves concentrated in degree 0, this definition of the groups $H^*(Z, K)$ extends the definition of the cohomology groups of a single sheaf given above to the case of bounded (below) complexes.

A quasi-isomorphism $K \to L$ induces isomorphisms on the cohomology, $H^i(U, K) \cong H^i(U, L)$ of any open set $U \subset Z$, and these isomorphisms are compatible with the maps induced by inclusions and with Mayer-Vietoris sequences.

3. The derived category. The *derived category* $D(Z)$ is a category whose objects are the complexes of sheaves, but whose morphisms have been cooked up in such a way that every quasi-isomorphism $S \to T$ becomes an isomorphism in $D(Z)$ (i.e., it has a unique inverse morphism). In this way, quite different complexes of sheaves that realize the same cohomology theory (such as the complex of singular cochains and the complex of differential forms on a $\mathcal{C}^\infty$ manifold) become isomorphic in $D(Z)$. The definition of the morphisms in the derived category is done by first identifying morphisms of complexes which are homotopic to each other, and then by formally adding inverses to quasi-isomorphisms. The second step is strongly reminiscent of the construction of the rational numbers as the field of fractions of the ring of integers, and the necessary calculus of fractions is made possible in view of the first step. An analogous notion is that of a *bounded derived category* $D^b(Z)$, where the objects are the bounded complexes of sheaves. The bounded derived category sits inside the derived category and the embedding $D^b(Z) \subseteq D(Z)$ is full, and similarly, for complexes bounded below (i.e., $\mathcal{H}^i(K) = 0, \forall i \ll 0$) and the corresponding category $D^+(Z) \subseteq D(Z)$, etc.

4. Derived functors. The main feature of the derived category is the possibility of defining derived functors. We discuss the case of cohomology and the case of the push-forward via a continuous map. If I is a bounded below complex of injective (flabby, or even fine) sheaves on Z, the cohomology $H^i(Z, I)$ is the cohomology of the complex of abelian groups

$$\cdots \longrightarrow \Gamma(Z, I^{i-1}) \longrightarrow \Gamma(Z, I^i) \longrightarrow \Gamma(Z, I^{i+1}) \longrightarrow \cdots,$$

which can be considered as an object, denoted $R\Gamma(Z, I)$, of the bounded below derived category of a point $D^+(pt)$. However, if the complex is not injective, as the example of the constant sheaf on a $\mathcal{C}^\infty$ manifold shows, this procedure gives the wrong answer, as the complexes of global sections of two quasi-isomorphic complexes are not necessarily quasi-isomorphic. Every bounded below complex K admits a bounded below injective resolution $K \to I$, unique up to a unique isomorphism in $D^+(Z)$. The complex of global sections $R\Gamma(Z, K) := \Gamma(Z, I)$ (a flabby resolution can be used as well and, if there is one, also a fine one) is well defined up to a unique isomorphism in the derived category $D^+(pt) \subseteq D(pt)$. For our limited purposes, note that we always work with bounded complexes whose resolutions can be chosen to be bounded; i.e., we can and do work within $D^b(Z)$, etc.

A similar construction arises when $f : W \to Z$ is a continuous mapping: if I is a bounded below complex of injective sheaves on W, then the push-forward complex $f_*(I)$ is a complex of sheaves on Z that satisfies

$$H^i(U, f_*(I)) \cong H^i(f^{-1}(U), I) \tag{5}$$

for any open set $U \subseteq Z$. However if a bounded below complex C on W is not injective, then (5) may fail, and $C \in D^+(W)$ should first be replaced by an injective resolution before pushing forward. The resulting complex of sheaves on Z is well defined up to a canonical isomorphism in $D^+(Z)$, is denoted Rf_*C and is called the (derived) *direct image* of C. Its cohomology sheaves are sheaves on Z, are denoted R^if_*C and are called the *i-th direct image sheaves.* Note that if f maps W to a point, then $Rf_*C = R\Gamma(W, C)$ and $R^if_*C = H^i(W, C)$.

When $f : W \to Z$ is a continuous map of locally compact spaces, a similar process, which starts with the functor *direct image with proper supports* $f_!$, yields the functor *derived direct image with proper supports* $Rf_! : D^+(W) \to D^+(Z)$. There is a map of functors $Rf_! \to Rf_*$, which is an isomorphism if f is proper. Under quite general hypotheses, always satisfied by algebraic maps of algebraic varieties, given a map $f : W \to Z$, there are the *inverse image* and *extraordinary inverse image* functors $f^*, f^! : D^b(Z) \to D^b(W)$. See §5.3 for a list of the properties of these four functors $Rf_*, Rf_!, f^*$ and $f^!$, as well as for their relation to Verdier duality.

5. Constructible sheaves. (See [87].) From now on, suppose Z is a complex algebraic variety. A subset $V \subset Z$ is *constructible* if it is obtained from a finite sequence of unions, intersections, or complements of algebraic subvarieties of Z. A *local system* on Z is a locally constant sheaf on Z with finite-dimensional stalks. A local system on Z corresponds to a finite-dimensional representation of the fundamental group of Z. A complex of sheaves K has *constructible cohomology sheaves* if there exists a decomposition $Z = \coprod_\alpha Z_\alpha$ into finitely many constructible subsets such that each of the cohomology sheaves $\mathcal{H}^i(K)$ is locally constant along each Z_α with finite-dimensional stalks. This implies that the limit

$$\mathcal{H}^i_x(K) := \lim_{\rightarrow} H^i(U_x, K) \tag{6}$$

is attained by any "regular" neighborhood U_x of the point x (for example, one may embed (locally) Z into a manifold and take $U_x = Z \cap B_\epsilon(x)$ to be the intersection of Z with a sufficiently small ball centered at x). This also implies that $H^i(Z, K)$ is finite dimensional. Constructibility prevents the cohomology sheaves from exhibiting Cantor-set-like behavior.

Most of the complexes of sheaves arising naturally from geometric constructions on varieties are bounded and have constructible cohomology sheaves.

From now on, in this survey, unless otherwise stated, bounded complexes with constructible cohomology sheaves are simply called *constructible complexes.*

The *constructible bounded derived category* $\mathcal{D}_Z$ is defined to be the full subcategory of the bounded derived category $D^b(Z)$ whose objects are the constructible complexes. This subcategory is stable under the Verdier duality functor; i.e., the dual of a constructible complex is a bounded constructible complex, it is stable under Hom, tensor products, vanishing and nearby cycle functors, and it is well-behaved with respect to the functors $Rf_*, Rf_!, f^*, f^!$ associated with an algebraic map $f : W \to Z$, i.e., $Rf_*, Rf_! : \mathcal{D}_W \to \mathcal{D}_Z$ and $f^*, f^! : \mathcal{D}_Z \to \mathcal{D}_W$.

6. Perverse sheaves, intersection complexes. A *perverse sheaf* is a constructible complex with certain restrictions (see §2.3) on the dimension of the support of its stalk cohomology and of its stalk cohomology with compact supports (i.e., the analogue with compact supports of (6)). These restrictions are called the *support* and *co-support conditions*, respectively.

Let $U \subset Z$ be a nonsingular Zariski open subset and let L be a local system on U. The *intersection complex* ([87]) $IC_Z(L)$ is a complex of sheaves on Z, which extends the complex $L[\dim Z]$ on U and is determined, up to unique isomorphism in $\mathcal{D}_Z$, by support and co-support conditions that are slightly stronger than the ones used to define perverse sheaves; see equations (12) and (13) in §2.1. In particular, intersection complexes are perverse sheaves. Up to a dimensional shift, the cohomology groups of the intersection complex $IC_Z(L)$ are the intersection cohomology groups of Z twisted by the system of local coefficients L: $H^i(Z, IC_Z(L)) = IH^{\dim Z + i}(Z, L)$.

The category of perverse sheaves is Abelian and Artinian (see §5.3): every perverse sheaf is an iterated extension of finitely many simple perverse sheaves. The simple perverse sheaves on Z are the intersection complexes $IC_Y(L)$ of irreducible subvarieties $Y \subset Z$ and irreducible local systems L defined on a nonsingular Zariski open subset of Y.

7. Perverse cohomology sheaves, perverse spectral sequence. The (ordinary) constructible sheaves, thought of as the constructible complexes which are concentrated in degree 0, form an Abelian full subcategory of the constructible derived category $\mathcal{D}_Z$. An object K of $\mathcal{D}_Z$ is isomorphic to an object of this subcategory if and only if $\mathcal{H}^i(K) = 0$ for every $i \neq 0$. There is a similar characterization of the category of perverse sheaves: every constructible complex $K \in \mathcal{D}_Z$ comes equipped with a canonical collection of perverse sheaves on Z, the *perverse cohomology sheaves* ${}^{\mathfrak{p}}\mathcal{H}^i(K)$, $i \in \mathbb{Z}$. The perverse sheaves are characterized, among the constructible complexes, by the property that ${}^{\mathfrak{p}}\mathcal{H}^i(K) = 0$ for every $i \neq 0$.

Just as there is the *Grothendieck spectral sequence*

$$E_2^{l,m} = H^l(Z, \mathcal{H}^m(K)) \Longrightarrow H^{l+m}(Z, K),$$

abutting to the *standard* (or Grothendieck) filtration, there is the *perverse spectral sequence*

$$E_2^{l,m} = H^l(Z, {}^{\mathfrak{p}}\mathcal{H}^m(K)) \Longrightarrow H^{l+m}(Z, K),$$

abutting to the perverse filtration, and similarly, for the cohomology groups with compact supports $H^*_c(Z, K)$.

Let $f : W \to Z$ be a map of varieties and $C \in \mathcal{D}_W$. We have $H^*(W, C) = H^*(Z, Rf_*C)$ and $H^*_c(W, C) = H^*_c(Z, Rf_!C)$. The *perverse Leray spectral sequence and filtration* for $H^*(W, C)$ and $H^*_c(W, C)$ are defined to be the perverse spectral sequence and filtrations for $H^*(Z, Rf_*C)$ and $H^*_c(Z, Rf_!C)$, respectively.

Remark 1.5.1. If U is a nonempty, nonsingular and pure dimensional open subset of Z on which all the cohomology sheaves $\mathcal{H}^i(K)$ are local systems, then the restrictions to U of ${}^{\mathfrak{p}}\mathcal{H}^m(K)$ and $\mathcal{H}^{m-\dim Z}(K)[\dim Z]$ coincide. In general, the two differ: in Example 1.8.4, we have ${}^{\mathfrak{p}}\mathcal{H}^0(Rf_*\mathbb{Q}_X[2]) \simeq IC_Y(R^1) \oplus T_\Sigma$. This illustrates the nontriviality of the notion of a perverse cohomology sheaf.

1.6. **Decomposition, semisimplicity and relative hard Lefschetz theorems.** Having dealt with some preliminaries on sheaves and derived categories, we now state

Theorem 1.6.1 (**Decomposition and semisimplicity theorems**). *Let $f : X \to Y$ be a proper map of complex algebraic varieties. There exists an isomorphism in the constructible bounded derived category $\mathcal{D}_Y$:*

$$Rf_*IC_X \simeq \bigoplus_{i \in \mathbb{Z}} {}^{\mathfrak{p}}\mathcal{H}^i(Rf_*IC_X)[-i]. \tag{7}$$

Furthermore, the perverse sheaves ${}^{\mathfrak{p}}\mathcal{H}^i(Rf_*IC_X)$ are semisimple; i.e., there is a decomposition into finitely many disjoint locally closed and nonsingular subvarieties $Y = \coprod S_\beta$ and a canonical decomposition into a direct sum of intersection complexes of semisimple local systems

$${}^{\mathfrak{p}}\mathcal{H}^i(Rf_*IC_X) \simeq \bigoplus_\beta IC_{\overline{S_\beta}}(L_\beta). \tag{8}$$

The decomposition theorem is usually understood to be the combination of (7) and (8), i.e., the existence of a finite collection of triples (Y_a, L_a, d_a) as in Theorem 1.4.1 such that we have a direct sum decomposition

$$Rf_*IC_X \simeq \bigoplus_a IC_{\overline{Y_a}}(L_a)[\dim X - \dim Y_a - d_a]. \tag{9}$$

Recalling that $IH^*(X) = H^{*-\dim X}(X, IC_X)$, the cohomological shifts in the formula above are chosen so that they match the ones of Theorem 1.4.1, which is in fact a consequence of (9). The local systems L_a are semisimple, and the collection of triples (Y_a, L_a, d_a) is essentially unique.

The direct sum decomposition (7) is finite and i ranges in the interval $[-r(f), r(f)]$, where $r(f)$ is the defect of semismallness of the map f (see §3.3.2, part 2, and [51]). In view of the properness of f and of the fact that IC_X is a self-dual complex (i.e., it coincides with its own dual), Poincaré-Verdier duality (cf. §5.8, duality exchanges) implies the existence of a canonical isomorphism

$$ {}^{\mathfrak{p}}\mathcal{H}^{-i}(f_* IC_X) \simeq {}^{\mathfrak{p}}\mathcal{H}^{i}(f_* IC_X)^{\vee}. \tag{10} $$

This important symmetry between the summands in (7) should not be confused with the somewhat deeper relative hard Lefschetz theorem, which is discussed below.

Remark 1.6.2 (**The splitting is not canonical)**. The splittings (7) and (9) are not uniquely determined. See Remark 1.4.2.

It seems worthwhile to list *some* important and immediate consequences of Theorem 1.6.1.

(1) The isomorphism (7) implies immediately that the perverse Leray spectral sequence
$$ E_2^{l,m} := H^l(Y, {}^{\mathfrak{p}}\mathcal{H}^m(Rf_* IC_X)) \Longrightarrow IH^{\dim X + l + m}(X, \mathbb{Q}) $$
is E_2-degenerate.

(2) If $f : X \to Y$ is a resolution of the singularities of a variety Y, i.e., if X is nonsingular and f is proper and an isomorphism away from a proper closed subset of Y, then one of the summands in (7) is IC_Y and we deduce that the intersection cohomology of Y is (noncanonically) a direct summand of the cohomology of any of its resolutions. Such resolutions exist, by a fundamental result of H. Hironaka.

(3) If $f : X \to Y$ is a proper submersion of nonsingular varieties, then, in view of Remark 1.5.1, the decomposition (9) can be rewritten as
$$ Rf_* \mathbb{Q}_X \simeq \bigoplus R^i f_* \mathbb{Q}_X[-i] $$
and one recovers Deligne's theorem ([56]) for families of projective manifolds (a weaker form of which is the E_2-degeneration of the Leray spectral sequence for such maps as stated in Theorem 1.2.1, part (1)). The semisimplicity statement of Theorem 1.6.1 corresponds then to Theorem 1.2.1, part (2).

As the name suggests, the *relative hard Lefschetz theorem* stated below is the relative version of the classical hard Lefschetz theorem seen in §1.1; i.e., it is a statement that occurs in connection with a map of varieties, which, when applied to the special case of the map of a projective manifold to a point, yields the classical hard Lefschetz theorem. The relative version is closely linked to the decomposition theorem as it expresses a symmetry among the summands in (7).

The symmetry in question arises when considering the operation of cupping with the first Chern class of *a hyperplane line bundle* on the domain of the map $f : X \to Y$. The hyperplane bundle on projective space is the holomorphic line bundle whose sections vanish precisely on linear hyperplanes. A hyperplane bundle on a quasi-projective variety X is the restriction to X of the hyperplane line bundle for some embedding $X \subseteq \mathbb{P}^N$.

The first Chern class of a line bundle η on X yields, for every $i \geq 0$, maps $\eta^i : Rf_*IC_X \to Rf_*IC_X[2i]$ and, by taking the perverse cohomology sheaves, we obtain maps of perverse sheaves $\eta^i : {}^{\mathfrak{p}}\mathcal{H}^{-i}(Rf_*IC_X) \longrightarrow {}^{\mathfrak{p}}\mathcal{H}^i(Rf_*IC_X)$.

Theorem 1.6.3 (**Relative hard Lefschetz theorem**). *Let $f : X \to Y$ be a proper map of varieties with X quasi-projective and let η be the first Chern class of a hyperplane line bundle on X. Then we have isomorphisms*

$$\eta^i : {}^{\mathfrak{p}}\mathcal{H}^{-i}(Rf_*IC_X) \xrightarrow{\simeq} {}^{\mathfrak{p}}\mathcal{H}^i(Rf_*IC_X). \tag{11}$$

If f is also a proper submersion, then we simply recover the classical hard Lefschetz theorem on the fibers of the map. As mentioned above, if we apply this result to the special case $f : X \to pt$, where X is a projective manifold, then we obtain the classical hard Lefschetz theorem. If X is a possibly singular projective variety, then we obtain the hard Lefschetz theorem in intersection cohomology (§1.4).

Remark 1.6.4. Theorems 1.6.1 and 1.6.3 also apply to $Rf_*IC_X(L)$ for certain classes of local systems L (see [9, 156]).

Example 1.6.5. Let $X = \mathbb{P}^1_{\mathbb{C}} \times \mathbb{C}$ and Y be the space obtained by collapsing the set $\mathbb{P}^1_{\mathbb{C}} \times \{o\}$ to a point. This is not a complex algebraic map, and (8) does not hold.

Example 1.6.6. Let $f : (\mathbb{C}^2 \setminus \{0\})/\mathbb{Z} =: X \to \mathbb{P}^1$ be the fibration in elliptic curves associated with a Hopf surface. Hopf surfaces are compact complex manifolds. Since $\pi_1(X) \simeq \mathbb{Z}$, we have $b_1(X) = 1$ so that X is not algebraic. In particular, though the map f is a proper holomorphic submersion, it is not an algebraic map and Deligne's theorem, and hence the decomposition theorem, does not apply. In fact, $Rf_*\mathbb{Q}_X$ does not split, for if it did, then $b_1(X) = 2$.

1.7. **Invariant cycle theorems.** The following theorem, in its local and global form, follows quite directly from the decomposition theorem. It generalizes previous results, which assume that X is smooth. For references, see the end of §1.10.

In a nutshell, the global invariant cycle Theorem 1.2.4 can be restated as asserting that if $f : X \to Y$ is a family of projective manifolds, then the monodromy invariants $H^*(F_y)^{\pi_1(Y,y)}$ on the cohomology of a fiber are precisely the image of the restriction map $H^*(X) \to H^*(F_y)$ from the total space of the family. (Clearly, the image of the restriction map is made of invariant classes, and the deep assertion is that every invariant class is global, i.e., it comes from X.) In view of the generalization given in Theorem 1.7.1 below, we conveniently restate this as the fact that the natural "edge" map

$$H^i(X) \longrightarrow H^0(Y, R^if_*\mathbb{Q}_X)$$

is surjective.

Theorem 1.7.1 (**Global and local invariant cycle theorems**). *Let $f : X \to Y$ be a proper map. Let $U \subseteq Y$ be a Zariski open subset on which the sheaf $R^if_*(IC_X)$ is locally constant. Then the following assertions hold.*

(1) (**Global**) *The natural restriction map*

$$IH^i(X) \longrightarrow H^0(U, R^if_*IC_X)$$

is surjective.

(2) (**Local**) *Let $u \in U$ and $B_u \subseteq U$ be the intersection with a sufficiently small Euclidean ball (chosen with respect to any local embedding of (Y, u) into a manifold) centered at u. Then the natural restriction/retraction map*

$$H^i(f^{-1}(u), IC_X) \simeq H^i(f^{-1}(B_u), IC_X) \longrightarrow H^0(B_u, R^i f_* IC_X)$$

is surjective.

1.8. **A few examples.** In this section we discuss the statement of the decomposition theorem in the following three examples: the resolution of singularities of a singular surface, the resolution of the affine cone over a projective nonsingular surface, and a fibration of a surface onto a curve. More details can be found in [53].

Example 1.8.1. Let $f : X \to Y$ be a resolution of the singularities of a singular surface Y. Assume that we have a single singular point $y \in Y$ with $f^{-1}(y) = E$ a finite union of curves on X. Since X is nonsingular, $IC_X = \mathbb{Q}_X[2]$ and we have an isomorphism

$$Rf_* \mathbb{Q}_X[2] \simeq IC_Y \oplus T,$$

where T is a skyscraper sheaf at y with stalk $T = H_2(E)$.

Example 1.8.2. Let $S \subseteq \mathbb{P}^N_{\mathbb{C}}$ be an embedded projective nonsingular surface and $Y \subseteq \mathbb{A}^{N+1}$ be the corresponding threefold affine cone over S. Let $f : X \to Y$ be the blowing up of Y at the vertex y. This is a resolution of the singularities of Y, it is an isomorphism outside the vertex of the cone and the fiber over the vertex is a copy of S. We have an isomorphism

$$Rf_* \mathbb{Q}_X[3] \simeq T_{-1}[1] \oplus (IC_Y \oplus T_0) \oplus T_1[-1],$$

where the T_j are skyscraper sheaves at y with stalks $T_1 \simeq T_{-1} \simeq H_4(S)$ and $T_0 \simeq H_3(S)$.

Example 1.8.3. Let $S \subseteq \mathbb{P}^3$ be the nonsingular quadric. The affine cone Y over S admits a resolution as in Example 1.8.2. This also admits resolutions $f : X' \to Y$, obtained by blowing up a plane passing through the vertex. In this case the exceptional fiber is isomorphic to $\mathbb{P}^1$ and we have $Rf_* \mathbb{Q}_{X'}[3] = IC_Y$.

Example 1.8.4. Let $f : X \to Y$ be a projective map with connected fibers from a smooth surface X onto a smooth curve Y. Let $\Sigma \subseteq Y$ be the finite set of critical values and let $U = Y \setminus \Sigma$ be its complement. The map f is a $\mathcal{C}^\infty$ fiber bundle over U with typical fiber a compact oriented surface of some fixed genus g. Let $R^1 = (R^1 f_* \mathbb{Q}_X)_{|U}$ be the rank $2g$ local system on U with stalk the first cohomology of the typical fiber. We have an isomorphism

$$Rf_* \mathbb{Q}_X[2] \simeq \mathbb{Q}_Y[2] \oplus (IC_Y(R^1) \oplus T_\Sigma) \oplus \mathbb{Q}_Y,$$

where T_Σ is a skyscraper sheaf over Σ with stalks $T_s \simeq H_2(f^{-1}(s))/\langle [f^{-1}(s)] \rangle$ at $s \in \Sigma$.

In all three examples the target space is a union $Y = U \amalg \Sigma$ and we have two corresponding types of summands. The summands of type T consist of classes which can be represented by cycles supported over the exceptional set Σ. This is precisely the kind of statement which lies at the heart of the decomposition theorem. There are classes which can be represented by intersection cohomology classes of local systems on Y and classes which can be represented by intersection cohomology classes of local systems supported over smaller strata, and the cohomology of X

is the direct sum of these two subspaces. Suggestively speaking, *it is as if the intersection cohomology relative to a stratum singled out precisely the classes which cannot be squeezed in the inverse image by f of a smaller stratum.*

1.9. **The decomposition theorem and mixed Hodge structures.** The proof of the hard Lefschetz theorem in intersection cohomology appears in [9]. Therefore, at that point in time, intersection cohomology was known to enjoy the two Lefschetz theorems and Poincaré duality ([9, 86, 87]). The question concerning a possible Hodge structure in intersection cohomology, as well as other Hodge-theoretic questions, was very natural at that juncture (cf. [9], p.165).

The work of M. Saito [156, 157] settled these issues completely with the use of mixed Hodge modules. The reader interested in the precise statements and generalizations is referred to Saito's papers (for brief summaries, see [70] and §3.2).

In this section, we summarize some of the mixed Hodge-theoretic properties of the intersection cohomology of complex quasi-projective varieties that we have reproved using classical Hodge theory (see §3.3).

The proofs can be found in [51, 54, 55, 45]. More precisely, the results for projective varieties and the maps between them (in this case, all Hodge structures are pure) are found in [51, 54] and the extension to quasi-projective varieties and the proper maps between them is found in [45], which builds heavily on [55].

Let us fix the setup. Let $f : X \to Y$ be a proper map of quasi-projective varieties. The intersection cohomology groups $IH^*(X)$ and $IH^*_c(X)$ are naturally filtered by the *perverse Leray filtration* P_*, where $P_pIH^*(X) \subseteq IH^*(X)$ and $P_pIH^*_c(X)$ are the images in cohomology and in cohomology with compact supports of the direct sum of terms i' with $i' \leq p$ in the decomposition theorem (7). Up to renumbering, this is the filtration abutment of the perverse Leray spectral sequence met in the crash course §1.5, and it can be defined and described geometrically regardless of the decomposition theorem (7); see §2.4. We abbreviate mixed Hodge structures as mHs.

(1) The intersection cohomology groups $IH^*(Y)$ and $IH^*_c(Y)$ carry natural mHs. If $f : X \to Y$ is a resolution of the singularities of Y, then these mHs are canonical subquotients of the mHs on $H^*(X)$ and on $H^*_c(X)$, respectively. If Y is a projective manifold, then the mHs is pure and it coincides with the classical one (Hodge decomposition). If Y is nonsingular, then the mHs coincide with Deligne's mHs on cohomology (see §5.2). The intersection bilinear pairing in intersection cohomology is compatible with the mHs; i.e., the resulting map $IH^{n-j}(Y) \longrightarrow (IH^{n+j}_c)^\vee(-n)$ is an isomorphism of mHs. The natural map $H^j(Y) \longrightarrow IH^j(Y)$ is a map of mHs; if Y is projective, then the kernel is the subspace W_{j-1} of classes of Deligne weight $\leq j-1$.

(2) If Y is a projective variety and η is a hyperplane line bundle on Y, then the hard Lefschetz theorem in intersection cohomology of §1.4 holds. In fact, the obvious transpositions from cohomology to intersection cohomology of the statements in §5.2, Theorem 5.2.1 hold.

(3) The subspaces P_p of the perverse Leray filtrations in $IH^*(X)$ and in $IH^*_c(X)$ are mixed Hodge substructures of the mHs mentioned in 1. The graded spaces of these filtrations (i.e., P_p/P_{p+1}) for $IH^*(X)$ and for $IH^*_c(X)$ inherit the natural quotient mHs, and they coincide (up to a shift in cohomological degree) with the cohomology and cohomology with compact supports of

the perverse cohomology sheaves ${}^{\mathfrak{p}}\mathcal{H}^{p}(Rf_*IC_X)$. We call these spaces the *perverse cohomology groups*.

(4) The splitting of the perverse cohomology groups associated with the canonical splitting (8) of the decomposition theorem takes place in the category of mHs.
(5) There exist splittings (7) for the decomposition theorem which induce isomorphisms of mHs in cohomology and in cohomology with compact supports. (Note that this statement is stronger than the one above: while these splittings take place in $IH^*(X)$ and in $IH^*_c(X)$, the previous ones take place in the perverse cohomology groups which are subquotients of $IH^*(X)$ and of $IH^*_c(X)$.)
(6) The mHs we introduce coincide with the ones obtained by M. Saito using mixed Hodge modules.

1.10. **Historical and other remarks.** In this section we offer a few remarks that describe the timeline for some of the results mentioned in this survey. We make no pretense to historical completeness. For an account of the development of intersection cohomology, see the historical remarks in [85] and the survey [120].

By the late 1920s, S. Lefschetz had "proofs" of the Lefschetz hyperplane and hard Lefschetz theorems in singular cohomology (see [121] for an interesting discussion of Lefschetz's proofs). Lefschetz's proof of the hard Lefschetz theorem is incomplete.

The Hodge decomposition theorem of cohomology classes into (p, q)-harmonic parts appears in W. Hodge's book [99]. This is where one also finds the first complete proof of the hard Lefschetz theorem (see also [178]). The proof of the (p, q) decomposition in [99] is not complete, and the missing analytical step was supplied by H. Weyl ([179]).

S.S. Chern gave a proof of the hard Lefschetz theorem in the 1950s (see [92]) which still relies on Hodge theory and exploits the action of $sl_2(\mathbb{C})$ on the differential forms on a Kähler manifold.

In the 1950s, R. Thom outlined a Morse-theoretic approach to the hyperplane theorem which was worked out in detail by A. Andreotti and T. Frankel [3] (see [144]) and by R. Bott [22].

The Hodge decomposition is the blueprint for the definition of pure and mixed Hodge structures given by P. Griffiths and by P. Deligne, respectively. The subject of how this decomposition varies in a family of projective manifolds and eventually degenerates has been studied, starting in the late 1960s, by P. Griffiths and his school. The degeneration of the Leray spectral sequence for families of projective manifolds was proved by P. Deligne in 1968.

In 1980, Deligne [62] gave a new proof and a vast generalization of the hard Lefschetz theorem by proving this result for varieties over finite fields and then inferring from this fact the result over the complex numbers. (One usually says that one "lifts the result from positive characteristic to characteristic zero"; see below.) In particular, the hard Lefschetz theorem is proved for varieties defined over an algebraically closed field. By a result of M. Artin, the Lefschetz hyperplane theorem also holds in this generality.

Poincaré duality for intersection cohomology is proved in [86]. The Lefschetz hyperplane theorem in intersection cohomology is proved in [87] and amplified in [85]. The hard Lefschetz theorem for the intersection cohomology of projective varieties is proved in [9]. In the 1980s, M. Saito ([156, 157]) proved that in the projective case these groups admit a pure Hodge structure (i.e., a (p, q)-Hodge decomposition), reproved that they satisfy the hard Lefschetz theorem and proved the Hodge-Riemann bilinear relations. In the 2000s, we reproved these results in [48, 51].

The decomposition theorem (3) for the intersection cohomology groups had been conjectured in 1980 by S. Gelfand and R. MacPherson. Note that they did not mention perverse sheaves. In fact, the decomposition theorem (9) only needs the notion of intersection cohomology in order to be formulated.

The decomposition, semisimplicity and relative hard Lefschetz theorems in §1.6 were proved by A. Beilinson, J. Bernstein, P. Deligne and O. Gabber in 1982 ([9]). They first proved it for proper maps of varieties and defined the algebraic closure of finite fields, and then they lifted the result to characteristic zero, i.e., for proper maps of complex algebraic varieties. In fact, they prove the result for the proper direct image of complexes of geometric origin (see Definition §3.1.14 in §3.1.5), and the intersection complex IC_X is a special and important example of a complex of geometric origin. They also proved the invariant cycle results summarized in Theorem 1.7.1. Finally, they proved the hard Lefschetz theorem (4) for intersection cohomology as a special case of their relative hard Lefschetz theorem. The equivariant versions of these results are proved in [14].

At that juncture, it was natural to ask: 1) for a proof of the decomposition theorem, semisimplicity and relative hard Lefschetz theorems for complex varieties that uses transcendental methods; about the existence of Hodge structures in intersection cohomology (pure in the compact case, mixed in the general case); 2) about Hodge-Riemann relations in intersection cohomology (in analogy with the ones for the singular cohomology of projective, or Kähler, manifolds; see Theorem 5.2.1 in §5.2); 3) about possible extensions of the decomposition theorem, etc. to intersection complexes with twisted coefficients underlying a polarized variation of pure Hodge structures; 4) about suitable extensions to quasi-projective varieties and mixed Hodge structures, and finally 5) about generalizations of all these results to the Kähler case (e.g., for proper holomorphic maps $f : X \to Y$, where X is a complex analytic space which admits a proper surjective and generically finite map onto it, e.g., a resolution of singularities, from a complex Kähler manifold).

All these questions have been answered in the work of M. Saito [156, 157] in the 1980s.

The case of IC_X (i.e., untwisted coefficients) and of quasi-projective varieties has been reproved by us using classical Hodge theory (see §1.9).

Finally, let us discuss the invariant cycle theorems. For families of projective manifolds, the global case was proved by P. Deligne, in [59], 4.1.1. The local case, conjectured and shown to hold for families of curves by P. Griffiths in [91], Conjecture. 8.1, was proved by P. Deligne in [62]. For Hodge-theoretic approaches to the local case, see [40, 168, 73, 96]. The "singular" case, i.e., Theorem 1.7.1, is proved in [9], p.164; see also [156].

2. Perverse sheaves

Perverse sheaves have become an important tool in the study of singular spaces as they enjoy many of the local and global properties of the constant sheaf that hold on nonsingular spaces, but that fail on singular ones. They are fundamental mathematical objects whose importance goes beyond their role in the proof of the decomposition theorem.

Here are some of the highlights of the theory of perverse sheaves. The reader can consult [9, 116, 68]. Recall that we are dealing with $\mathbb{Q}$-coefficients and with middle-perversity only. We refer to §1.5 and §5 for more details and amplifications.

Historically, perverse sheaves arose naturally from the theory of D-modules, i.e., the sheaf-theoretic reformulation of linear systems of partial differential equations: The "solution sheaf" of a holonomic D-module with regular singularities is a perverse sheaf, and this (Riemann-Hilbert correspondence) defines a functor from the category of holonomic D-modules with regular singularities to perverse sheaves.

Even though the D-modules side of the story is a necessary complement to the more topological-oriented approach presented here, for lack of competence, we do not treat it in this paper. A partial list of references is [20, 13, 113, 114, 115, 139, 140, 15].

Let Y be a complex algebraic variety. Like the category of constructible sheaves, the category $\mathcal{P}_Y$ of perverse sheaves is a full subcategory of the constructible derived category $\mathcal{D}_Y$. The category $\mathcal{P}_Y$ is Abelian, Noetherian and Artinian (i.e., every perverse sheaf is a finite iterated extension of simple perverse sheaves). The simple perverse sheaves on Y are the intersection complexes $IC_W(L)$ associated with an irreducible and closed subvariety $W \subseteq Y$ and an irreducible local system L (on a Zariski-dense open nonsingular subvariety of W). Since $\mathcal{P}_Y$ is an Abelian category, any morphism in $\mathcal{P}_Y$ admits a ("perverse") kernel and ("perverse") cokernel. Given a complex $K \in \mathcal{D}_Y$, there are the ("perverse") cohomology sheaves ${}^{\mathfrak{p}}\mathcal{H}^i(K) \in \mathcal{P}_Y$. A theorem of A. Beilinson's states that the bounded derived category of $\mathcal{P}_Y$ is again $\mathcal{D}_Y$. Many operations work better in the category of perverse sheaves than in the category of sheaves, e.g., the duality and vanishing cycles functors preserve perverse sheaves. The Lefschetz hyperplane theorem holds for perverse sheaves. Specialization over a curve takes perverse sheaves to perverse sheaves. The intersection cohomology of a projective variety satisfies the Hodge-Lefschetz theorems and Poincaré duality.

2.1. **Intersection cohomology.** The intersection cohomology complex of a complex algebraic variety Y is a special case of a perverse sheaf, and every perverse sheaf is a finite iterated extension of intersection complexes. It seems appropriate to start a discussion of perverse sheaves with this most important example.

Given a complex n-dimensional algebraic variety Y and a local system L on a nonsingular Zariski-dense open subvariety $U \subseteq Y$, there exists a constructible complex of sheaves $IC_Y(L) \in \mathcal{D}_Y$, unique up to canonical isomorphism in $\mathcal{D}_Y$, such that $IC(L)_{|U} \cong L$ and:

$$\dim \left\{ y \in Y | \ \mathcal{H}^i_y(IC(L)) \neq 0 \right\} < -i, \text{ if } i > -n, \tag{12}$$

$$\dim \left\{ y \in Y | \ \mathcal{H}^i_{c,y}(IC(L)) \neq 0 \right\} < i, \text{ if } i < n, \tag{13}$$

where, for any complex S of sheaves,

$$\mathcal{H}^i_{c,y}(S) = \lim_{\leftarrow} H^i_c(U_y, S)$$

is the local *compactly supported* cohomology at x. (As explained in the "crash course" §1.5, if S is constructible, then the above limit is attained by any regular neighborhood U_y of y.) The intersection complex $IC_Y(L)$ is sometimes called the *intermediate extension* of L. Its (shifted) cohomology is the intersection cohomology of Y with coefficients in L, i.e., $IH^{n+*}(Y, L) := H^*(Y, IC_Y(L))$. The reader can consult [86, 87] and [19, 68].

Even though intersection cohomology lacks functoriality with respect to algebraic maps (however, see [5]), the intersection cohomology groups of projective varieties enjoy the same properties of Hodge-Lefschetz-Poincaré type as the singular cohomology of projective manifolds. Poincaré duality takes the form $IH^k(Y) \simeq IH^{2n-k}(Y)^\vee$ and follows formally from the canonical isomorphism $IC_Y \simeq IC_Y^\vee$ stemming from Poincaré-Verdier duality; in particular, there is a nondegenerate geometric intersection pairing

$$IH^i(Y) \times IH^{2n-i}(Y) \longrightarrow \mathbb{Q}, \qquad (a, b) \longmapsto a \cdot b;$$

on the other hand there is no cup product. As to the other properties, i.e., the two Lefschetz theorems, the Hodge decomposition and the Hodge-Riemann bilinear relations, see §1.9 and §3.3.

2.2. Examples of intersection cohomology.

Example 2.2.1. Let $E^{n-1} \subseteq \mathbb{P}^N$ be a projective manifold, $Y^n \subseteq \mathbb{A}^{N+1}$ be the associated affine cone. The link $\mathcal{L}$ of Y at the vertex o of the cone, i.e., the intersection of Y with a sufficiently small Euclidean sphere centered at o, is an oriented compact smooth manifold of real dimension $2n-1$ and is an S^1-fibration over E. The cohomology groups of $\mathcal{L}$ are

$$H_{2n-1-j}(\mathcal{L}) = H^j(\mathcal{L}) = P^j(E), \;\; 0 \leq j \leq n-1,$$
$$H^{n-1+j}(\mathcal{L}) = P^{n-j}(E), \;\; 0 \leq j \leq n,$$

where $P^j(E) \subseteq H^j(E)$ is the subspace of primitive vectors for the given embedding of E, i.e., the kernel of cupping with the appropriate power of the first Chern class of $\mathcal{O}_E(-E)$. The Poincaré intersection form on $H^*(\mathcal{L})$ is nondegenerate, as usual, and also because of the Hodge-Riemann bilinear relations (38) on E.

The intersection cohomology groups of Y are

$$IH^j(Y) \simeq P^j(E) = H^j(\mathcal{L}), \;\; 0 \leq j \leq n-1, \qquad IH^j(Y) = 0, \;\; n \leq j \leq 2n.$$

The intersection cohomology with compact supports of Y are

$$IH_c^{2n-j}(Y) \simeq H_j(\mathcal{L}), \;\; 0 \leq j \leq n-1, \qquad IH_c^{2n-j}(Y) = 0, \;\; n \leq j \leq 2n.$$

We thus see that, in this case, the Poincaré duality isomorphism $IH^j(Y) \simeq IH_c^{2n-j}(Y)^\vee$ stems from the classical Poincaré duality on the link.

In the remaining part of this section, we complement some examples of intersection complexes and groups with some further information expressed using the language of perverse sheaves which we discuss in the next few sections.

Example 2.2.2. Let Y be the projective cone over a nonsingular curve $C \subseteq \mathbb{P}^N$ of genus g. The cohomology groups are

$$H^0(Y) = \mathbb{Q}, \quad H^1(Y) = 0, \quad H^2(Y) = \mathbb{Q}, \quad H^3(Y) = \mathbb{Q}^{2g}, \quad H^4(Y) = \mathbb{Q}.$$

The intersection cohomology groups are:

$$IH^0(Y) = \mathbb{Q}, \quad IH^1(Y) = \mathbb{Q}^{2g}, \quad IH^2(Y) = \mathbb{Q}, \quad IH^3(Y) = \mathbb{Q}^{2g}, \quad IH^4(Y) = \mathbb{Q}.$$

Note the failure of Poincaré duality in cohomology and its restoration via intersection cohomology. There is a canonical resolution $f : X \to Y$ of the singularities of Y obtained by blowing up the vertex of Y. The decomposition theorem yields a splitting exact sequence of perverse sheaves on Y:

$$0 \longrightarrow IC_Y \longrightarrow f_*\mathbb{Q}_X[2] \longrightarrow H^2(C)[0] \longrightarrow 0.$$

Example 2.2.3. We now revisit Example 1.6.5. Let $f : X \to Y$ be the space obtained by contracting to a point $v \in Y$, the zero section $C \subseteq \mathbb{P}^1 \times C =: X$. This example is analogous to the one in Example 2.2.2, except that Y is not a complex algebraic variety. The cohomology groups are

$$H^0(Y) = \mathbb{Q}, \quad H^1(Y) = 0, \quad H^2(Y) = \mathbb{Q}, \quad H_3(Y) = \mathbb{Q}^{2g}, \quad H_4(Y) = \mathbb{Q}.$$

The stratified space Y has strata of even codimension, and we can define its intersection complex, etc. The intersection cohomology groups are:

$$IH^0(Y) = \mathbb{Q}, \quad IH^1(Y) = \mathbb{Q}^{2g}, \quad IH_2(Y) = 0, \quad IH^3(Y) = \mathbb{Q}^{2g}, \quad IH_4(Y) = \mathbb{Q}.$$

Note the failure of Poincaré duality in cohomology and its restoration via intersection cohomology. There is a natural epimorphism of perverse sheaves $\tau : f_*\mathbb{Q}_X[2] \longrightarrow H^2(C)[0]$. There are nonsplitting exact sequences in $\mathcal{P}_Y$:

$$0 \longrightarrow \operatorname{Ker}\tau \longrightarrow f_*\mathbb{Q}_X[2] \longrightarrow H^2(C)[0] \longrightarrow 0,$$
$$0 \longrightarrow IC_Y \longrightarrow \operatorname{Ker}\tau \longrightarrow \mathbb{Q}_v[0] \longrightarrow 0.$$

The complex $f_*\mathbb{Q}_X[2]$ is a perverse sheaf on Y obtained by a two-step extension procedure involving intersection complexes (two of which are skyscraper sheaves). The intersection cohomology complexes IC_Y and $\mathbb{Q}_v$ of Y and $v \in Y$ appear in this process, but not as direct summands. The conclusion of the decomposition theorem does not hold for this map f.

Example 2.2.4. Let Y be the projective cone over the quadric $\mathbb{P}^1 \times \mathbb{P}^1 \simeq Q \subseteq \mathbb{P}^3$. The odd cohomology is trivial. The even cohomology is as follows:

$$H^0(Y) = 0, \quad H^2(Y) = \mathbb{Q}, \quad H^4(Y) = \mathbb{Q}^2, \quad H^6(Y) = \mathbb{Q}.$$

The intersection cohomology groups are the same as the cohomology groups, except that $IH^2(Y) = \mathbb{Q}^2$. Note the failure of Poincaré duality in homology and its restoration via intersection homology. There are at least two different and interesting resolutions of the singularities of Y: the ordinary blowup of the vertex $o \in Y$, $f : X \to Y$ which has fiber $f^{-1}(o) \simeq Q$, and the blowup of any line on the cone through the origin $f' : X' \to Y$ which has fiber ${f'}^{-1}(o) \simeq \mathbb{P}^1$. The decomposition theorem yields (cf. Example 1.8.2)

$$f_*\mathbb{Q}_X[3] = IC_Y \oplus \mathbb{Q}_o[1] \oplus \mathbb{Q}_o[-1], \qquad f'_*\mathbb{Q}_{X'}[3] = IC_Y.$$

Example 2.2.5. Let E be the rank two local system on the punctured complex line $\mathbb{C}^*$ defined by the automorphism of $e_1 \mapsto e_1$, $e_2 \mapsto e_1 + e_2$. It fits into the nontrivial extension

$$0 \longrightarrow \mathbb{Q}_{\mathbb{C}^*} \longrightarrow E \xrightarrow{\phi} \mathbb{Q}_{\mathbb{C}^*} \longrightarrow 0.$$

Note that E is self-dual. If we shift this extension by [1], then we get a nonsplit exact sequence of perverse sheaves in $\mathcal{P}_{\mathbb{C}^*}$. Let $j : \mathbb{C}^* \to \mathbb{C}$ be the open immersion. The complex $IC_{\mathbb{C}}(E) = R^0 j_* E[1]$ is a single sheaf in cohomological degree -1 with generic stalk $\mathbb{Q}^2$ and stalk $\mathbb{Q}$ at the origin $0 \in \mathbb{C}$. In fact, this stalk is given by the space of invariants which is spanned by the single vector e_1. We remark, in passing, that given any local system L on $\mathbb{C}^*$, we have that $IC_{\mathbb{C}}(L) = R^0 j_* L[1]$. There is the monic map $\mathbb{Q}_{\mathbb{C}}[1] \to IC_{\mathbb{C}}(E)$. The cokernel K' is the nontrivial extension, unique since $\mathrm{Hom}(\mathbb{Q}_{\mathbb{C}}, \mathbb{Q}_{\{0\}}) = \mathbb{Q}$ is one dimensional,

$$0 \longrightarrow \mathbb{Q}_{\{0\}} \longrightarrow K' \longrightarrow \mathbb{Q}_{\mathbb{C}}[1] \longrightarrow 0.$$

Note that while the perverse sheaf $IC_{\mathbb{C}}(E)$, being an intermediate extension (§2.7), has no subobjects and no quotients supported at $\{0\}$, it has a subquotient supported at $\{0\}$, namely the perverse sheaf $\mathbb{Q}_{\{0\}}$. We shall meet this example again later in Example 2.7.1, in the context of the nonexactness of the intermediate extension functor.

Example 2.2.6. Let $\Delta \subseteq \mathbb{C}^n$ be the subset $\Delta = \{(x_1, \ldots, x_n) \in \mathbb{C}^n : \prod x_i = 0\}$. The datum of n commuting endomorphisms $T_1, \ldots, T_n$ of a $\mathbb{Q}$-vector space V defines a local system L on $(\mathbb{C}^*)^n = \mathbb{C}^n \setminus \Delta$ whose stalk at some base point p is identified with V, and T_i is the monodromy along the path "turning around the divisor $x_i = 0$." The vector space V has a natural structure of a $\mathbb{Z}^n = \pi_1((\mathbb{C}^*)^n, p)$-module. The complex which computes the group cohomology $H^\bullet(\mathbb{Z}^n, V)$ of V can be described as follows: Let $e_1, \ldots, e_n$ be the canonical basis of $\mathbb{Q}^n$, and, for $I = (i_0, \ldots, i_k)$, set $e_I = e_{i_0} \wedge \ldots \wedge e_{i_k}$. Define a complex (C, d) by setting

$$C^k = \bigoplus_{0 < i_0 < \ldots < i_k < n} V \otimes e_I, \qquad d(v \otimes e_I) = \sum N_i(v) \otimes e_i \wedge e_I,$$

with $N_i := T_i - I$. Since $(\mathbb{C}^*)^n$ has no higher homotopy groups, we have the quasi-isomorphism $(j_* L)_0 \stackrel{qis}{\simeq} (C, d)$. Let

$$\widetilde{C^k} = \bigoplus_{0 < i_0 < \ldots < i_k < n} N_I V \otimes e_I,$$

where $N_I := N_{i_0} \circ \ldots \circ N_{i_k}$. It is clear that $(\widetilde{C}, d)$ is a subcomplex of (C, d). There exists a natural isomorphism $(IC_{\mathbb{C}^n}(L))_0 \simeq (\widetilde{C}, d)$. The particularly important case in which L underlies a polarized variation of Hodge structures has been investigated in depth in [38] and [117].

2.3. **Definition and first properties of perverse sheaves.** Let $K \in \mathcal{D}_Y$ be a constructible complex on the variety Y. Recall that the support of a sheaf is the closure of the set of points where the sheaf has nontrivial stalks. We say that K satisfies the *support condition* if

$$\dim \{\mathrm{Supp}\, \mathcal{H}^{-i}(K)\} \leq i, \quad \forall\, i \in \mathbb{Z}.$$

We say that K satisfies the *co-support condition* if the Verdier dual $K^\vee$ (§5.3) satisfies the conditions of support.

By Verdier duality, we have $\mathcal{H}^i_y(K^\vee) \simeq \mathcal{H}^{-i}_{c,y}(K)^\vee$, so that we may write the support and co-support conditions as follows:

$$\dim \left\{ y \in Y \middle|\ \mathcal{H}^i_y(K) \neq 0 \right\} \leq -i,\ \forall\ i \in \mathbb{Z}, \tag{14}$$

$$\dim \left\{ y \in Y \middle|\ \mathcal{H}^i_{c,y}(K) \neq 0 \right\} \leq i,\ \forall i \in \mathbb{Z}. \tag{15}$$

Definition 2.3.1. A *perverse sheaf* on Y is a constructible complex K in $\mathcal{D}_Y$ that satisfies the conditions of support and co-support. The category $\mathcal{P}_Y$ of perverse sheaves is the full subcategory of $\mathcal{D}_Y$ whose objects are the perverse sheaves.

A complex K is perverse iff $K^\vee$ is perverse. The defining conditions of intersection complexes in §2.1 are a stricter version of the support and co-support conditions given above. It follows that intersection complexes are special perverse sheaves.

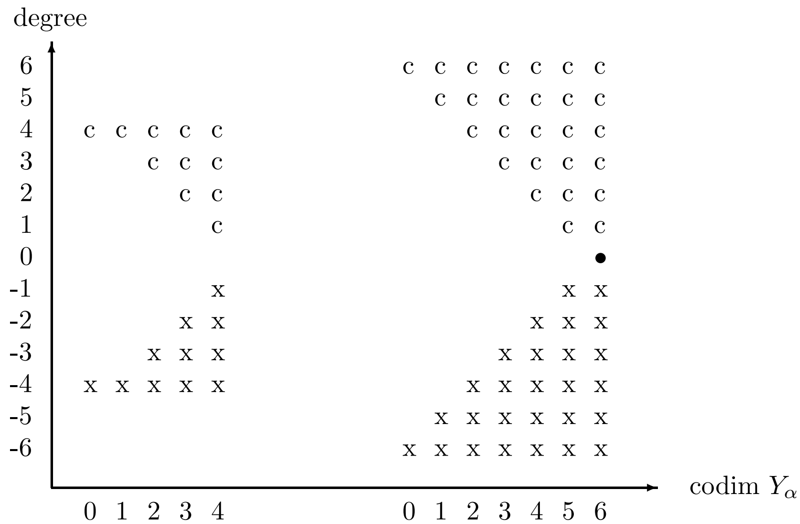

FIGURE 1. Support conditions for IC (left) and for a perverse sheaf (right)

Figure 1 illustrates the support and co-support conditions for intersection cohomology on a variety of dimension 4 (left) and a perverse sheaf on a variety of dimension 6 (right). The symbol "c" means that compactly supported stalk cohomology can be nonzero at that place, while the symbol "x" means that stalk cohomology can be nonzero at that place. Note that the $\bullet$ symbol shows that, for a perverse sheaf, there is a place at which both compactly supported and ordinary cohomology can be nonzero. As explained in §5.7.1, the natural map $\mathcal{H}^i_{c,y}(-) \to \mathcal{H}^i_y(-)$ governs the splitting behaviour of the perverse sheaf.

Denote by $\mathcal{P}_Y$ the full subcategory of $\mathcal{D}_Y$ whose objects are perverse sheaves. Denote by ${}^{\mathfrak{p}}\mathcal{D}^{\leq 0}_Y$ (${}^{\mathfrak{p}}\mathcal{D}^{\geq 0}_Y$, resp.) the full subcategory of $\mathcal{D}_Y$ with objects the complexes satisfying the conditions of support (co-support, resp.). Clearly, ${}^{\mathfrak{p}}\mathcal{D}^{\leq 0}_Y \cap {}^{\mathfrak{p}}\mathcal{D}^{\geq 0}_Y = \mathcal{P}_Y$. These data give rise to the *middle perversity t-structure* on $\mathcal{D}_Y$ (see §5.3).

Theorem 2.3.2. *The datum of the conditions of (co-)support together with the associated full subcategories* $({}^{\mathfrak{p}}\mathcal{D}^{\leq 0}_Y, {}^{\mathfrak{p}}\mathcal{D}^{\geq 0}_Y)$ *yields a t-structure on* $\mathcal{D}_Y$*, called the middle perversity t-structure, with heart* ${}^{\mathfrak{p}}\mathcal{D}^{\leq 0}_Y \cap {}^{\mathfrak{p}}\mathcal{D}^{\geq 0}_Y$*, the category of perverse sheaves* $\mathcal{P}_Y$*.*

The resulting truncation and cohomology functors are denoted, for every $i \in \mathbb{Z}$:

$$ {}^{\mathfrak{p}}\tau_{\leq i} : \mathcal{D}_Y \longrightarrow {}^{\mathfrak{p}}\mathcal{D}^{\leq i}_Y, \quad {}^{\mathfrak{p}}\tau_{\geq i} : \mathcal{D}_Y \longrightarrow {}^{\mathfrak{p}}\mathcal{D}^{\geq i}_Y, $$

$$ {}^{\mathfrak{p}}\mathcal{H}^0 = {}^{\mathfrak{p}}\tau_{\geq 0}\, {}^{\mathfrak{p}}\tau_{\leq 0}, \; {}^{\mathfrak{p}}\mathcal{H}^i = {}^{\mathfrak{p}}\mathcal{H}^0 \circ [i] \; : \mathcal{D}_Y \longrightarrow \mathcal{P}_Y. $$

In particular, any complex $K \in \mathcal{D}_Y$ has "perverse cohomology sheaves" ${}^{\mathfrak{p}}\mathcal{H}^i(K) \in \mathcal{P}_Y$.

The key point in the proof is to show the existence of ${}^{\mathfrak{p}}\tau_{\geq 0}$ and ${}^{\mathfrak{p}}\tau_{\leq 0}$. The construction of these perverse truncation functors involves only the four functors $f^*, f_*, f_!, f^!$ for open and closed immersions and standard truncation. See [9] or [116]. Complete and brief summaries can be found in [52, 53].

Middle-perversity is well-behaved with respect to Verdier duality: the Verdier duality functor $D : \mathcal{P}_Y \to \mathcal{P}_Y$ is an equivalence and we have canonical isomorphisms

$$ {}^{\mathfrak{p}}\mathcal{H}^i \circ D \simeq D \circ {}^{\mathfrak{p}}\mathcal{H}^{-i}, \qquad {}^{\mathfrak{p}}\tau_{\leq i} \circ D \simeq D \circ {}^{\mathfrak{p}}\tau_{\geq -i}, \qquad {}^{\mathfrak{p}}\tau_{\geq i} \circ D \simeq D \circ {}^{\mathfrak{p}}\tau_{\leq -i}. $$

It is not difficult to show, by using the perverse cohomology functors (see §2.5), that $\mathcal{P}_Y$ is an Abelian category. As is customary when dealing with Abelian categories, when we say that $A \subseteq B$ (A is included in B), we mean that there is a monomorphism $A \to B$. The Abelian category $\mathcal{P}_Y$ is Noetherian (i.e., every increasing sequence of perverse subsheaves of a perverse sheaf must stabilize) so that, by Verdier duality, it is also Artinian (i.e., every decreasing sequence stabilizes). The category of constructible sheaves is Abelian and Noetherian, but not Artinian.

A. Beilinson [7] has proved that, remarkably, the bounded derived category of perverse sheaves $D^b(\mathcal{P}_Y)$ is equivalent to $\mathcal{D}_Y$. There is a second, also remarkable, equivalence due to M. Nori. Let $D^b(CS_Y)$ be the bounded derived category of the category of constructible sheaves on Y (the objects are bounded complexes of constructible sheaves). There is a natural inclusion of categories $D^b(CS_Y) \subseteq \mathcal{D}_Y$ (recall that the objects of $\mathcal{D}_Y$ are bounded complexes of sheaves whose cohomology sheaves are constructible). M. Nori [153] has proved that the inclusion $D^b(CS_Y) \subseteq \mathcal{D}_Y$ is an equivalence of categories. This is a striking instance of the phenomenon that a category can arise as a derived category in fundamentally different ways: $\mathcal{D}_Y \simeq D^b(\mathcal{P}_Y) \simeq D^b(CS_Y)$.

Perverse sheaves, just like ordinary sheaves, form a stack ([9], 3.2); i.e., suitably compatible systems of perverse sheaves can be glued to form a single perverse sheaf, and similarly for compatible systems of morphisms of perverse sheaves. This is not the case for the objects and morphisms of $\mathcal{D}_Y$; e.g., a nontrivial extension of vector bundles yields a nonzero morphism in the derived category that restricts to zero on the open sets of a suitable open covering, i.e., where the extension restricts to trivial extensions.

Example 2.3.3. Let Y be a point. The standard and perverse t-structures coincide. A complex $K \in \mathcal{D}_{pt}$ is perverse iff it is isomorphic in $\mathcal{D}_{pt}$ to a complex concentrated in degree zero iff $\mathcal{H}^j(K) = 0$ for every $j \neq 0$.

Example 2.3.4. If Y is a variety of dimension n, then the complex $\mathbb{Q}_Y[n]$ trivially satisfies the conditions of support. If $n = \dim Y = 0, 1$, then $\mathbb{Q}_Y[n]$ is perverse. On a surface Y with isolated singularities, $\mathbb{Q}_Y[2]$ is perverse iff the singularity is unibranch, e.g., if the surface is normal. If (Y, y) is a germ of a threefold isolated singularity, then $\mathbb{Q}_Y[3]$ is perverse iff the singularity is unibranch and $H^1(Y \setminus y) = 0$.

Example 2.3.5. The direct image $f_*\mathbb{Q}_X[n]$ via a proper semismall map $f : X \to Y$, where X is a nonsingular n-dimensional nonsingular variety, is perverse (see Proposition 4.2.1); e.g., a generically finite map of surfaces is semismall. For an interesting, nonsemisimple, perverse sheaf arising from a nonalgebraic semismall map, see Example 2.2.3.

Perverse sheaves are stable under the following functors: intermediate extension, nearby and vanishing cycle (see §5.5).

Let $i : Z \to Y$ be the closed immersion of a subvariety of Y. One has the functor $i_* : \mathcal{P}_Z \to \mathcal{P}_Y$. This functor is fully faithful; i.e., it induces a bijection on the Hom-sets. It is customary, e.g. in the statement of the decomposition theorem, to drop the symbol i_*.

Let Z be an irreducible closed subvariety of Y and L be an irreducible (i.e., without trivial local subsystems, i.e., simple in the category of local systems) local system on a nonempty Zariski open subvariety of the regular part Z_{reg} of Z. Recall that a simple object in an abelian category is one without trivial subobjects. The complex $IC_Z(L)$ is a simple object of the category $\mathcal{P}_Y$. Conversely, every simple object of $\mathcal{P}_Y$ has this form. This follows from the following proposition [9], which yields a direct proof of the fact that $\mathcal{P}_Y$ is Artinian.

Recall that by an inclusion $A \subseteq B$, we mean the existence of a monomorphism $A \to B$, so that by a chain of inclusions, we mean a chain of monomorphisms. The following caveat may be useful, as it points out that the usual set-theoretic intuition about injectivity and surjectivity may be misleading when dealing with perverse sheaves, or with Abelian categories in general. Let $j : \mathbb{C}^* \to \mathbb{C}$ be the open immersion. We have a natural injection of sheaves $j_! \mathbb{Q}_{\mathbb{C}^*} \to \mathbb{Q}_{\mathbb{C}}$. On the other hand, one can see that the induced map of perverse sheaves $j_! \mathbb{Q}_{\mathbb{C}^*}[1] \to \mathbb{Q}_{\mathbb{C}}[1]$ is not a monomorphism; in fact, it is an epimorphism.

Proposition 2.3.6 (Composition series). *Let $P \in \mathcal{P}_Y$. There exists a finite decreasing filtration*

$$P = Q_1 \supseteq Q_2 \supseteq \ldots \supseteq Q_\lambda = 0,$$

where the quotients Q_i/Q_{i-1} are simple perverse sheaves on Y. Every simple perverse sheaf is of the form $IC_{\overline{Z}}(L)$, where $Z \subseteq Y$ is an irreducible and nonsingular subvariety and L is an irreducible local system on Z.

As usual, in this kind of situation, e.g., the Jordan-Hölder theorem for finite groups, the filtration is not unique, but the constituents of P, i.e., the nontrivial simple quotients, and their multiplicities are uniquely determined.

2.4. **The perverse filtration.** The theory of t-structures coupled with Verdier's formalism of spectral objects (cf. [58], Appendix), endows the cohomology groups $H^*(Y,K)$ with the canonical *perverse filtration* P defined by $P^p H^*(Y,K) := \mathrm{Im}\,\{H^*(Y, {}^{\mathfrak{p}}\tau_{\leq -p}K) \to H^*(Y,K)\}$, which is, up to renumbering, the abutment of the *perverse spectral sequence* $H^p(Y, {}^{\mathfrak{p}}\mathcal{H}^q(K)) \Longrightarrow H^*(Y,K)$. See §1.5, (7). A similar result holds for cohomology with compact supports.

In [55], we give a geometric description of the perverse filtration on the cohomology and on the cohomology with compact supports of a constructible complex on a quasi-projective variety. The paper [45] gives an alternative proof with the applications to mixed Hodge theory mentioned in §1.9; the paper [46] proves similar results for the standard filtration on cohomology with compact supports.

The description is in terms of restriction to generic hyperplane sections and it is somewhat unexpected, especially if one views the constructions leading to perverse sheaves as transcendental and hyperplane sections as more algebro-geometric. If $f : X \to Y$ is a map of quasi-projective varieties and $C \in \mathcal{D}_X$, then our results yield a similar geometric description of the perverse Leray filtration on $H^*(X,C)$ and on $H^*_c(X,C)$ induced by the map f.

We now describe the perverse filtration on the cohomology groups $H^*(Y,K)$ when Y is *affine*. Let $Y_* = \{Y \supseteq Y_{-1} \supseteq \ldots \supseteq Y_{-n}\}$ be a sequence of closed subvarieties; we call this data an n-flag. Basic sheaf theory endows $H^*(Y,K)$ with the flag filtration F, abutment of the spectral sequence associated with the filtration by closed subsets $Y_* \subseteq Y$: $E_1^{p,q} = H^{p+q}(Y_p, Y_{p-1}, K_{|Y_p}) \Longrightarrow H^*(Y,K)$. We have $F^pH^*(Y,K) = \mathrm{Ker}\,\{H^*(Y,K) \to H^*(Y_{p-1}, K_{|Y_{p-1}})\}$. For an arbitrary n-flag, the perverse and flag filtrations are unrelated. If Y is *affine* of dimension n and the n-flag is obtained using n hyperplane sections in sufficiently general position, then

$$P^pH^j(Y,K) = F^{p+j}H^j(Y,K). \tag{16}$$

2.5. **Perverse cohomology.** The functor ${}^{\mathfrak{p}}\mathcal{H}^0 : \mathcal{D}_Y \to \mathcal{P}_Y$ sends a complex K to its iterated truncation ${}^{\mathfrak{p}}\tau_{\leq 0}\,{}^{\mathfrak{p}}\tau_{\geq 0}K$. This functor is cohomological. In particular, given a distinguished triangle $K' \to K \to K'' \to K'[1]$, one obtains a long exact sequence

$$\cdots \longrightarrow {}^{\mathfrak{p}}\mathcal{H}^j(K') \longrightarrow {}^{\mathfrak{p}}\mathcal{H}^j(K) \longrightarrow {}^{\mathfrak{p}}\mathcal{H}^j(K'') \longrightarrow {}^{\mathfrak{p}}\mathcal{H}^{j+1}(K') \longrightarrow \cdots .$$

Kernels and cokernels in $\mathcal{P}_Y$ can be seen via perverse cohomology. Let $f : K \to K'$ be an arrow in $\mathcal{P}_Y$. View it in $\mathcal{D}_Y$, cone it and obtain a distinguished triangle

$$K \xrightarrow{f} K' \longrightarrow \mathrm{Cone}(f) \longrightarrow K[1].$$

Take the associated long exact sequence of perverse cohomology

$$0 \longrightarrow {}^{\mathfrak{p}}\mathcal{H}^{-1}(\mathrm{Cone}(f)) \longrightarrow K \xrightarrow{f} K' \longrightarrow {}^{\mathfrak{p}}\mathcal{H}^0(\mathrm{Cone}(f)) \longrightarrow 0.$$

One verifies that $\mathcal{P}_Y$ is abelian by setting

$$\mathrm{Ker}\, f := {}^{\mathfrak{p}}\mathcal{H}^{-1}(\mathrm{Cone}(f)), \qquad \mathrm{Coker}\, f := {}^{\mathfrak{p}}\mathcal{H}^0(\mathrm{Cone}(f)).$$

Example 2.5.1. Consider the natural map $a : \mathbb{Q}_Y[n] \to IC_Y$. Since $\mathbb{Q}_Y[n] \in {}^p\mathcal{D}_Y^{\leq 0}$, and IC_Y does not admit nontrivial subquotients, the long exact sequence of perverse cohomology sheaves yields the following short exact sequences:

$${}^{\mathfrak{p}}\mathcal{H}^{l<0}(\mathrm{Cone}(a)) \simeq {}^{\mathfrak{p}}\mathcal{H}^{l<0}(\mathbb{Q}_Y[n]),$$
$$0 \to {}^{\mathfrak{p}}\mathcal{H}^0(\mathrm{Cone}(a)) \longrightarrow {}^{\mathfrak{p}}\mathcal{H}^0(\mathbb{Q}_Y[n]) \longrightarrow IC_Y \to 0.$$

If Y is a normal surface, then $\mathbb{Q}_Y[2]$ is perverse and we are left with the short exact sequences in $\mathcal{P}_Y$:

$$0 \longrightarrow {}^{\mathfrak{p}}\mathcal{H}^0(\mathrm{Cone}(a)) \longrightarrow \mathbb{Q}_Y[2] \xrightarrow{a} IC_Y \longrightarrow 0.$$

By taking the long exact sequence associated with $\mathcal{H}^j$, one sees that ${}^{\mathfrak{p}}\mathcal{H}^0(\mathrm{Cone}(a))$ reduces to a skyscraper sheaf supported at the singular points of Y in cohomological degree zero and a stalk computed by the cohomology of the link of Y at y : $\mathcal{H}^{-1}(IC_Y)_y = H^1(\mathcal{L}_y)$. Note that, in general, the short exact sequence does not split; i.e., $\mathbb{Q}_X[2]$ is not necessarily a semisimple perverse sheaf.

Example 2.5.2 (Blowing up with smooth centers). Let $X \to Y$ be the blowing up of a manifold Y along a codimension $r+1$ submanifold $Z \subseteq Y$. One has an isomorphism in $\mathcal{D}_Y$:

$$f_*\mathbb{Q}_X \simeq \mathbb{Q}_Y[0] \oplus \bigoplus_{j=1}^{r} \mathbb{Q}_Z[-2j].$$

If $r+1$ is odd (the even case is analogous and left to the reader), then

$$ {}^{\mathfrak{p}}\mathcal{H}^0(f_*\mathbb{Q}_X[n]) = \mathbb{Q}_Y[n], \qquad {}^{\mathfrak{p}}\mathcal{H}^j(f_*\mathbb{Q}_X[n]) = \mathbb{Q}_Z[\dim Z], \ \ 0 < |j| \leq r/2. $$

We have three sets of summands, i.e., $(j > 0, j = 0, j < 0)$. Poincaré-Verdier duality exchanges the first and third sets and fixes the second. The relative hard Lefschetz theorem identifies the first set with the third.

Example 2.5.3 (**Families of projective manifolds**). Let $f : X \to Y$ be a family of d-dimensional projective manifolds and let $n := \dim X$. Theorem 1.2.1 is the cohomological consequence of a stronger sheaf-theoretic result (cf. [56]): there is a direct sum decomposition in $\mathcal{D}_Y$:

$$ f_*\mathbb{Q}_X \ \simeq \ \bigoplus_{j=0}^{2d} R^j f_*\mathbb{Q}_X[-j]. $$

We have

$$ {}^{\mathfrak{p}}\mathcal{H}^j(f_*\mathbb{Q}_X[n]) = R^{d+j} f_*\mathbb{Q}_X[\dim Y], \quad j \in \mathbb{Z}. $$

If we apply Poincaré duality and the hard Lefschetz theorem to the fibers of f, we obtain the following isomorphisms (where the second one is obtained by cupping with $c_1(H)^j$, where H is a hyperplane bundle on X):

$$ {}^{\mathfrak{p}}\mathcal{H}^j(f_*\mathbb{Q}_X[n]) \simeq {}^{\mathfrak{p}}\mathcal{H}^{-j}(f_*\mathbb{Q}_X[n])^\vee, \quad \forall j \in \mathbb{Z}, $$

$$ {}^{\mathfrak{p}}\mathcal{H}^{-j}(f_*\mathbb{Q}_X[n]) \simeq {}^{\mathfrak{p}}\mathcal{H}^j(f_*\mathbb{Q}_X[n]), \quad \forall j \geq 0. $$

2.6. **t-exactness and the Lefschetz hyperplane theorem.** The following prototypical Lefschetz-type result is a consequence of the left t-exactness of affine maps (cf. §5.3).

Proposition 2.6.1. *Let $f : X \to Y$ be a proper map, $C \in {}^{\mathfrak{p}}\mathcal{D}^{\geq 0}_X$. Let $Z \subseteq X$ be a closed subvariety, $U := X \setminus Z$. The diagram of maps*

$$ \begin{tikzcd} U \arrow[r, "j"] \arrow[dr, "h"'] & X \arrow[d, "f"] & Z \arrow[l, "i"'] \arrow[dl, "g"] \\ & Y & \end{tikzcd} $$

is commutative. Assume that h is affine. Then

$$ {}^{\mathfrak{p}}\mathcal{H}^j(f_*C) \longrightarrow {}^{\mathfrak{p}}\mathcal{H}^j(g_*i^*C) $$

is an isomorphism for $j \leq -2$ and is a monomorphism for $j = -1$.

Proof. By applying $f_! = f_*$ to the distinguished triangle $j_!j^!C \to C \to i_*i^*C \xrightarrow{[1]}$ we get the distinguished triangle

$$ h_!j^*C \longrightarrow f_*C \longrightarrow g_*i^*C \xrightarrow{[1]} \cdot $$

Since h is affine, $h_!$ is left t-exact, so that

$$ {}^{\mathfrak{p}}\mathcal{H}^j(h_!j^*C) = 0 \qquad \forall j < 0. $$

The result follows by taking the long exact sequence of perverse cohomology. $\square$

Taking $C = IC_Y$ and f to be the map to a point, and observing that $i^*IC_Y[-1] = IC_Z$, gives the following Lefschetz hyperplane theorem ([87], Theorem 7.1) in intersection cohomology.

Theorem 2.6.2 (Lefschetz hyperplane theorem for intersection cohomology). *Let Y be an irreducible projective variety of dimension n and $Z \subseteq Y$ be a general hyperplane section. The restriction*

$$IH^l(Y) \longrightarrow IH^l(Z)$$

is an isomorphism for $l \leq n-2$ and monic for $l = n-1$.

Remark 2.6.3. One has the dual result for the Gysin map in the positive cohomological degree range. Similar conclusions hold for the cohomology groups of any perverse sheaf on Y (see [7], Lemma 3.3).

Another related special case of Proposition 2.6.1, used in [9] and in [51] as one step towards the proof of the relative hard Lefschetz theorem, arises as follows. Let $\mathbb{P} \supseteq X' \to Y'$ be a proper map, let $\mathbb{P}^\vee$ be the dual projective space to $\mathbb{P}$, whose points parametrize the hyperplanes in $\mathbb{P}$, let $Z := \{(x', H) \mid x' \in H\} \subseteq X := X' \times \mathbb{P}^\vee$ be the universal hyperplane section, $Y := Y' \times \mathbb{P}^\vee$. In this case, $IC_Z = i^* IC_X[-1]$. We have

Theorem 2.6.4 (Relative Lefschetz hyperplane theorem). *The natural map*

$$^{\mathfrak{p}}\mathcal{H}^j(f_* IC_X) \longrightarrow {}^{\mathfrak{p}}\mathcal{H}^{j+1}(g_* IC_Z)$$

is an isomorphism for $j \leq -2$ and monic for $j = -1$.

2.7. **Intermediate extensions.** A standard reference is [9]. Let $j : U \to Y$ be a locally closed embedding Y and $i : \overline{U} \backslash U =: Z \to Y$. Given a perverse sheaf Q on U, the *intermediate extension* (often called the "middle extension") $j_{!*} : \mathcal{P}_U \to \mathcal{P}_{\overline{U}}$ is a simple operation that produces distinguished perverse extensions to $\overline{U}$ and hence to Y.

Intersection complexes are intermediate extensions: let L be a local system on a nonsingular open and dense subvariety U of an irreducible d-dimensional variety Y; then $IC_Y(L) = j_{!*}L[d]$.

Let $Q \in \mathcal{P}_U$. The natural map $j_! Q \longrightarrow j_* Q$ induces the natural map in perverse cohomology $a : {}^{\mathfrak{p}}\mathcal{H}^0(j_! Q) \to {}^{\mathfrak{p}}\mathcal{H}^0(j_* Q)$. The *intermediate extension* of $Q \in \mathcal{P}_U$ is the perverse sheaf

$$j_{!*}Q := \operatorname{Im}(a) \in \mathcal{P}_{\overline{U}} \subseteq \mathcal{P}_Y.$$

There is the following canonical factorization in the abelian categories $\mathcal{P}_{\overline{U}} \subseteq \mathcal{P}_Y$:

$$^{\mathfrak{p}}\mathcal{H}^0(j_! Q) \xrightarrow{epic} j_{!*}Q \xrightarrow{monic} {}^{\mathfrak{p}}\mathcal{H}^0(j_* Q).$$

The intermediate extension $j_{!*}Q$ admits several useful characterizations. For example:

(1) it is the unique extension of $Q \in \mathcal{P}_U$ to $\mathcal{P}_{\overline{U}} \subseteq \mathcal{P}_Y$ with neither subobjects nor quotients supported on Z;
(2) it is the unique extension $\widetilde{Q}$ of $Q \in \mathcal{P}_U$ to $\mathcal{P}_{\overline{U}} \subseteq \mathcal{P}_Y$ such that $i^*\widetilde{Q} \in {}^{\mathfrak{p}}\mathcal{D}_Z^{\leq -1}$ and $i^!\widetilde{Q} \in {}^{\mathfrak{p}}\mathcal{D}_Z^{\geq 1}$.

There are an additional characterization of and a precise formula involving standard truncation and derived push-forwards for the intermediate extension functor (cf. [9], 2.1.9 and 2.1.11) both of which involve stratifications. This formula implies that: i) if $j : U \to Y$ is an open immersion of irreducible varieties and U is nonsingular of dimension d, then $j_{!*}L[d]$ is canonically isomorphic to $IC_Y(L)$; ii) if Y is a nonsingular curve, then $j_{!*}L[1] = IC_Y(L) = R^0 j_* L[1]$.

We leave to the reader the task to formulate in precise terms and verify that the intermediate extension of an intermediate extension is an intermediate extension.

An intersection cohomology complex, being an intermediate extension, does not admit subobjects or quotients supported on proper subvarieties of its support.

The intermediate extension functor $j_{!*} : \mathcal{P}_U \to \mathcal{P}_Y$ is not exact in a funny way. Let $0 \to P \stackrel{a}{\to} Q \stackrel{b}{\to} R \to 0$ be exact in $\mathcal{P}_U$. Recall that $j_!$ is right t-exact and that j_* is left t-exact. We have the following display with exact rows.

$$\begin{tikzcd}
\cdots \arrow[r] & j_!P \arrow[r] \arrow[d, "epic"] & j_!Q \arrow[r, "epic"] \arrow[d, "epic"] & j_!R \arrow[r] \arrow[d, "epic"] & 0 \\
& j_{!*}P \arrow[d, "monic"] & j_{!*}Q \arrow[d, "monic"] & j_{!*}R \arrow[d, "monic"] & \\
0 \arrow[r] & j_*P \arrow[r, "monic"] & j_*Q \arrow[r] & j_*R \arrow[r] & \cdots
\end{tikzcd}$$

It is a simple diagram-chasing exercise to complete the middle row functorially with a necessarily monic $j_{!*}(a)$ and a necessarily epic $j_{!*}(b)$. It follows that the intermediate extension functor preserves monic and epic maps. What fails is the exactness "in the middle": in general, $\operatorname{Ker} j_{!*}(b) / \operatorname{Im} j_{!*}(a) \neq 0$.

Example 2.7.1. Let $E[1]$ be the perverse sheaf on $\mathbb{C}^*$ discussed in Example 2.2.5; recall that it fits in the nonsplit short exact sequence of perverse sheaves:

$$0 \longrightarrow \mathbb{Q}[1] \stackrel{a}{\longrightarrow} E[1] \stackrel{b}{\longrightarrow} \mathbb{Q}[1] \longrightarrow 0.$$

Let $j : \mathbb{C}^* \to \mathbb{C}$ be the open immersion. We have the following commutative diagram of perverse sheaves with exact top and bottom rows.

$$\begin{tikzcd}
0 \arrow[r] & j_!\mathbb{Q}[1] \arrow[r] \arrow[d, "epic"] & j_!E[1] \arrow[r] \arrow[d, "epic"] & j_!\mathbb{Q}[1] \arrow[r] \arrow[d, "epic"] & 0 \\
& \mathbb{Q}_{\mathbb{C}}[1] \arrow[r, "monic", "j_{!*}(a)"'] \arrow[d, "monic"] & R^0 j_* E[1] \arrow[r, "epic", "j_{!*}(b)"'] \arrow[d, "monic"] & \mathbb{Q}_{\mathbb{C}}[1] \arrow[d, "monic"] & \\
0 \arrow[r] & j_*\mathbb{Q}[1] \arrow[r] & j_*E[1] \arrow[r] & j_*\mathbb{Q}[1] \arrow[r] & 0.
\end{tikzcd}$$

The middle row, i.e., the one of middle extensions, is not exact in the middle. In fact, inspection of the stalks at the origin yields the nonexact sequence

$$0 \longrightarrow \mathbb{Q} \stackrel{\simeq}{\longrightarrow} \mathbb{Q} \stackrel{0}{\longrightarrow} \mathbb{Q} \longrightarrow 0.$$

This failure prohibits exactness in the middle. The inclusion $\operatorname{Im} j_{!*}(a) \subseteq \operatorname{Ker} j_{!*}(b)$ is strict: $K := \operatorname{Ker} j_{!*}(b)$ is the unique nontrivial extension, $\operatorname{Hom}(\mathbb{Q}_{\{0\}}, \mathbb{Q}_{\mathbb{C}}[2]) = \mathbb{Q}$,

$$0 \longrightarrow \mathbb{Q}_{\mathbb{C}}[1] \longrightarrow K \longrightarrow \mathbb{Q}_{\{0\}} \longrightarrow 0.$$

The reader can check, e.g., using the self-duality of E, that $K^\vee = K'$ (K' as in Ex. 2.2.5).

Property 1, characterizing intermediate extensions, is used in the construction of composition series for perverse sheaves in Proposition 2.3.6. It follows that $j_{!*}Q$ is a simple perverse sheaf on Y iff Q is a simple perverse sheaf on U.

Example 2.7.2 (Intersection cohomology complexes with different supports). Let $Z_1, Z_2 \subseteq Y$ be irreducible closed subvarieties with $Z_1 \neq Z_2$ (note that we are allowing $Z_1 \cap Z_2 \neq \emptyset$). Let $IC_{Z_i}(L_i)$, $i = 1, 2$, be intersection cohomology complexes. Then (cf. [87], Theorem 3.5)

$$\mathrm{Hom}(IC_{Z_1}(L_1), IC_{Z_2}(L_2)) = 0.$$

In fact, the kernel (cokernel, resp.) of any such map would have to be either zero, or supported on Z_1 (Z_2, resp.), in which case it is easy to complete the construction by virtue of characterization 1 given above.

Here is a nice application of what has been said above. Let $f : X \to Y$ be a proper and semismall map of irreducible proper varieties; see §4.2. The decomposition theorem yields a (canonical in this case) splitting

$$f_* IC_X = \bigoplus IC_{Z_a}(L_a).$$

Poincaré duality on IC_X yields a canonical isomorphism $e : f_* IC_X \simeq (f_* IC_X)^\vee$, which, by Example 2.7.2, is a direct sum map. It follows that the direct summands $IH^*(Z_a, L_a) \subseteq IH^*(X)$ are mutually orthogonal with respect to the Poincaré pairing.

3. Three approaches to the decomposition theorem

3.1. **The proof of Beilinson, Bernstein, Deligne and Gabber.** The original proof [9] of the decomposition theorem for proper maps of complex algebraic varieties uses in an essential way the language of the étale cohomology of l-adic sheaves and the arithmetic properties of varieties defined over finite fields.

In this section we try to introduce the reader to some of the main ideas in [9].

Let us first give a very brief and rough summary of these ideas. The theory of weights, i.e., of the eigenvalues of the Frobenius automorphisms on the stalks of l-adic sheaves on varieties defined over finite fields, leads to the notion of pure complexes. There are many pure complexes: O. Gabber proved that the intersection cohomology complex of a variety is a pure perverse sheaf. The push-forward via a proper map of algebraic varieties defined over a finite field of a pure complex is a pure complex. After passing to an algebraic closure of a finite field, a pure complex splits as a direct sum of shifted intersection complexes with coefficients in lisse irreducible sheaves (a lisse l-adic sheaf is the l-adic analogue of a local system). We thus obtain the decomposition theorem for the proper push-forward of a pure complex, e.g., the intersection complex of a variety, at least after passing to the algebraic closure of the finite field.

Associated with a map of complex algebraic varieties there are companion maps of varieties defined over finite fields. There is the class of constructible complexes of geometric origin over complex varieties. A complex of geometric origin over a complex variety admits l-adic counterparts on the companion varieties defined over the finite fields. The intersection complex is of geometric origin.

The decomposition result over the algebraic closures of the finite fields is shown to imply the analogous result in (i.e., it lifts to) the complex algebraic setting and we finally obtain the decomposition theorem in the complex setting.

The idea that results over finite fields can be used to prove results over the complex numbers is rooted in the classical result that a system of rational polynomial equations has a solution over an algebraic number field if it has a solution modulo an infinite number of prime numbers.

There are several appearances of this idea in the literature, often in connection with a beautiful discovery. Here are a few: P. Deligne and D. Mumford's proof [66] that the moduli space of curves of a given genus is irreducible in any characteristic, S. Mori's proof [147] of Hartshorne's conjecture, P. Deligne and L. Illusie's algebraic proof [64] of the Kodaira vanishing theorem and of the degeneration of the Hodge to de Rham spectral sequence (see the nice survey [105]).

A precursor of the techniques used in lifting the decomposition theorem from finite fields to the complex numbers is P. Deligne's proof ([62]) of the hard Lefschetz theorem.

We do not discuss further the "lifting" technique, and we refer the reader to [9], §6.

The goal of the remaining part of this section is to introduce the reader to constructible $\overline{\mathbb{Q}}_l$-sheaves (§3.1.1), weights, pure complexes and their structure (§3.1.2, §3.1.3), to discuss the decomposition, semisimplicity and hard Lefschetz theorems in the context of pure complexes over finite fields and over their algebraic closures (§3.1.4), and to state the decomposition theorem, etc., for complexes of geometric origin on complex algebraic varieties (§3.1.5).

We hope that our stating separately the results over finite fields, over their algebraic closures and over the complex numbers may help the reader better understand the whole picture and perhaps justifies the tediousness of these repetitions.

Let us fix some notation. A variety over a field is a separated scheme of finite type over that field. For a quick summary on quasi-projective varieties (which is all we need here), see §5.1. Let $\mathbb{F}_q$ be a finite field, let $\mathbb{F}$ be a fixed algebraic closure of $\mathbb{F}_q$ and let $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$ be the Galois group. This group is profinite, isomorphic to the profinite completion of $\mathbb{Z}$, and it admits as topological generator the geometric Frobenius $Fr := \varphi^{-1}$, where $\varphi : \mathbb{F} \to \mathbb{F}$, $t \mapsto t^q$ is the arithmetic Frobenius (see Remark 3.1.1). Let $l \neq \mathrm{char}\, \mathbb{F}_q$ be a fixed prime number, let $\mathbb{Z}_l$ be the ring of l-adic integers, i.e., the projective limit of the system $\mathbb{Z}/l^n\mathbb{Z}$ (abbreviated by $\mathbb{Z}/l^n$), let $\mathbb{Q}_l$ be the l-adic numbers, i.e., the quotient field of $\mathbb{Z}_l$, and let $\overline{\mathbb{Q}}_l$ be a fixed algebraic closure of $\mathbb{Q}_l$. Recall that $\mathbb{Z}_l$ is uncountable and that $\overline{\mathbb{Q}}_l \simeq \mathbb{C}$, noncanonically.

3.1.1. *Constructible $\overline{\mathbb{Q}}_l$-sheaves.* Let X_0 be an algebraic variety defined over a finite field $\mathbb{F}_q$. We refer to [9, 62], and to the introductory [75], §12, for the definitions of the category and $D^b_c(X_0, \overline{\mathbb{Q}}_l)$ of constructible complexes of $\overline{\mathbb{Q}}_l$-sheaves. These categories are stable under the usual operations $f^*, f_*, f_!, f^!$, derived Hom and tensor product, duality and vanishing and nearby cycles. With some homological restrictions on Tor groups, the standard and the middle perverse t-structure are also defined, and one obtains the category $\mathcal{P}(X_0, \overline{\mathbb{Q}}_l)$ of perverse sheaves on X_0. If X is the $\mathbb{F}$-variety obtained from X_0 by extending the scalars to $\mathbb{F}$, then we obtain in the same way the categories $D^b_c(X, \overline{\mathbb{Q}}_l)$ and $\mathcal{P}(X, \overline{\mathbb{Q}}_l)$ which are also stable under the usual operations mentioned above.

The construction of these categories and functors and the verification of their fundamental properties requires a massive background (a large part of Grothendieck et al. S.G.A. seminars is devoted to this task) and has led P. Deligne to complete the proof of the Weil Conjectures ([61]), one of the crowning achievements of 20$^{\text{th}}$ century mathematics.

For the purpose of this survey, let us just say that we will mostly think of $D^b_c(X_0, \overline{\mathbb{Q}}_l)$, etc., by analogy with the perhaps more geometric constructible derived categories $\mathcal{D}_X$ associated with complex varieties. There is one important difference: the action of the Frobenius automorphism.

3.1.2. *Weights and purity.* In positive characteristic, the étale cohomology of algebraic varieties presents a feature that is absent in characteristic zero: the eigenvalues of Frobenius, i.e., *weights*.

Let X_0 be a variety over the finite field $\mathbb{F}_q$. Suppression of the index $-_0$ denotes extension of scalars from $\mathbb{F}_q$ to $\mathbb{F} = \overline{\mathbb{F}}_q$. For example, if F_0 is a $\overline{\mathbb{Q}}_l$-sheaf on X_0, then we denote its pull-back to X by F.

To give a $\overline{\mathbb{Q}}_l$-sheaf F_0 on the one-point variety $\operatorname{Spec}\mathbb{F}_q$ is equivalent to giving a finite-dimensional continuous $\overline{\mathbb{Q}}_l$-representation of the Galois group $\operatorname{Gal}(\mathbb{F}/\mathbb{F}_q)$. The pull-back F to $\operatorname{Spec}\mathbb{F}$ is the sheaf given by the underlying $\overline{\mathbb{Q}}_l$-vector space of the representation (i.e., we "forget" the representation; this is because the Galois group $\operatorname{Gal}(\mathbb{F}/\mathbb{F})$ is trivial). This is called the *stalk* of F_0 at the point.

It is important to keep in mind that the sheaf F_0 on $\operatorname{Spec}\mathbb{F}_q$ must be thought of as the pair given by the vector space *and* the representation, while its pull-back F to $\operatorname{Spec}\mathbb{F}$ is just the datum of the vector space. This partially explains why the decomposition theorem holds over the algebraic closure $\mathbb{F}$, but not necessarily over the finite field $\mathbb{F}_q$, where the splittings have to be compatible with the Frobenius action.

There are restrictions on the representations arising in this context: e.g., in the case of a $\mathbb{Q}_l$-sheaf of rank one on $\operatorname{Spec}\mathbb{F}_q$, keeping in mind that the Galois group is compact, continuity implies that $Fr \in \operatorname{Gal}(\mathbb{F}/\mathbb{F}_q)$ must act by units in $\mathbb{Z}_l \subseteq \overline{\mathbb{Q}}_l$.

Remark 3.1.1. It is often useful to keep in mind the following roughly approximated picture when thinking about the extension $\mathbb{F}/\mathbb{F}_q$: think of the one-point variety $\operatorname{Spec}\mathbb{F}_q$ as being a circle S^1; think of the extension $\mathbb{F}/\mathbb{F}_q$ as being the universal covering space $\mathbb{R} \to S^1$ with deck group given by translations by integers; think of the Galois group as being the deck group; given an l-adic sheaf F_0 on the one-point variety $\operatorname{Spec}\mathbb{F}_q$, think of the action of Frobenius on the stalk of this sheaf as the $\mathbb{Z}$-action on a sheaf on $\mathbb{R}$, the pull-back of a sheaf on S^1.

For every $n \geq 1$, the finite set $X_0(\mathbb{F}_{q^n})$ of closed points in X_0 which are defined over the degree n extension $\mathbb{F}_q \subseteq \mathbb{F}_{q^n}$ is precisely the set of closed points (we are using the Zariski topology) of X which are fixed under the action of the n-th iterate, $Fr^n : X \to X$, of the geometric Frobenius $Fr : X \to X$. Recall that if, for example, X_0 is defined by a system of polynomials $\{P_i(T)\}$ in $\mathbb{F}_q[T_1, \ldots, T_N]$, then a closed point of X_0 defined over $\mathbb{F}_{q^n}$ can be identified with an N-tuple $(a_1, \ldots, a_N) \in \mathbb{F}^N_{q^n}$ which is a solution of the system of polynomial equations $P_i(T) = 0$.

Let $x \in X_0(\mathbb{F}_{q^n})$ be such an Fr^n-fixed point. The $\overline{\mathbb{Q}}_l$-sheaf F_0 restricted to x has stalk the $\overline{\mathbb{Q}}_l$-vector space F_x on which Fr^n acts as an automorphism.

Definition 3.1.2 (Punctually pure). The $\overline{\mathbb{Q}}_l$-sheaf F_0 on X_0 is *punctually pure of weight w* ($w \in \mathbb{Z}$) if, for every $n \geq 1$ and every $x \in X_0(\mathbb{F}_{q^n})$, the eigenvalues of the action of Fr^n on F_x are algebraic numbers such that all of their complex algebraic conjugates have absolute value $q^{n\,w/2}$.

For example, on $\operatorname{Spec}\mathbb{F}_q$, the sheaf $\overline{\mathbb{Q}}_l$ has weight 0, while the Tate-twisted $\overline{\mathbb{Q}}_l(1)$ has weight -2. If X_0 is a nonsingular projective curve of genus g, then the étale cohomology group $H^1_{et}(X_0, \overline{\mathbb{Q}}_l)$ can be viewed as an l-adic sheaf on $\operatorname{Spec} \mathbb{F}_q$ with weight 1.

The eigenvalues of Frobenius are naturally elements of $\overline{\mathbb{Q}}_l$. While $\overline{\mathbb{Q}}_l \simeq \mathbb{C}$, there is no natural isomorphism between them. However, since $\mathbb{Q} \subseteq \overline{\mathbb{Q}}_l$, it makes sense to request that the eigenvalues are algebraic numbers (i.e., their being algebraic is independent of the choice of an isomorphism $\overline{\mathbb{Q}}_l \simeq \mathbb{C}$). Once a number is algebraic, the set of its algebraic conjugates is well defined independently of a choice of an isomorphism $\overline{\mathbb{Q}}_l \simeq \mathbb{C}$, and this renders meaningful the request on the absolute values. This is a strong request: $1+\sqrt{2}$ and $1-\sqrt{2}$ are algebraic conjugates; however, they have different absolute values.

Definition 3.1.3 (**Mixed sheaf, weights**). A $\overline{\mathbb{Q}}_l$-sheaf F_0 on X_0 is *mixed* if it admits a finite filtration with punctually pure successive quotients. The *weights* of a mixed F_0 are the weights of the nonzero quotients.

Definition 3.1.4 (**Mixed and pure complexes**). The category $D^b_m(X_0, \overline{\mathbb{Q}}_l)$ of *mixed complexes* is the full subcategory of $D^b_c(X_0, \overline{\mathbb{Q}}_l)$ given by those complexes whose cohomology sheaves are mixed. A complex $K_0 \in D^b_m(X_0, \overline{\mathbb{Q}}_l)$ is *pure of weight w* if the cohomology sheaves $\mathcal{H}^i(K_0)$ are punctually pure of weights $\leq w+i$ and the same is true for its Verdier dual $K_0^\vee$.

The following theorem is proved in [9] (see §3.3.1 and §6.2.3) and is a key step towards the proof of the decomposition theorem given in [9]. Note that the special case when X_0 is nonsingular and projective and $Y_0 = \operatorname{Spec} \mathbb{F}_q$ yields a proof of the main result in [61], i.e., the completion of the proof of the Weil conjectures.

Theorem 3.1.5 (**Purity for proper maps or relative Weil conjectures**). *Let K_0 be pure of weight w and $f_0 : X_0 \to Y_0$ be a proper map of $\mathbb{F}_q$-varieties. Then $f_{0*}K_0$ is pure of weight w.*

3.1.3. *The structure of pure complexes.* In this section we state the Gabber purity theorem and discuss the special splitting features of pure complexes.

The following result of O. Gabber [78] was never published. A proof appears in [9], Corollaire 5.4.3, and it is summarized in [29]. This result makes it clear that the class of pure complexes contains many geometrically relevant objects.

Recall that lisse $\overline{\mathbb{Q}}_l$-sheaves are the $\overline{\mathbb{Q}}_l$-analogues of local systems in the classical topology.

Theorem 3.1.6 (**Gabber purity theorem**). *The intersection cohomology complex IC_{X_0} of a connected pure d-dimensional variety X_0 is pure of weight d. More generally, if L is a pure lisse $\overline{\mathbb{Q}}_l$-sheaf of weight w on a connected, pure d-dimensional subvariety $j : Z_0 \to X_0$, then $IC_{\overline{Z}_0}(L) := j_{!*}L[d]$ is a pure perverse sheaf of weight $w+d$.*

The following result ([9], Corollaire 5.3.4) generalizes Gabber's Purity theorem and is another key step in the proof in [9] of the decomposition, semisimplicity and relative hard Lefschetz theorems over the complex numbers.

Theorem 3.1.7 (**Mixed and simple is pure**). *Let $P_0 \in \mathcal{P}_m(X_0, \overline{\mathbb{Q}}_l)$ be a simple mixed perverse $\overline{\mathbb{Q}}_l$-sheaf. Then P_0 is pure.*

The following theorem summarizes the basic splitting properties of pure complexes. The proofs can be found in [9], Théorèmes 5.4.1, 5.4.5 and 5.4.6, and Corollaire 5.3.8.

Theorem 3.1.8 (Purity and decompositions). (1) *Let $K_0 \in D^b_m(X_0, \overline{\mathbb{Q}}_l)$ be pure of weight w. Each ${}^{\mathfrak{p}}\mathcal{H}^i(K_0)$ is a pure perverse sheaf of weight $w+i$. There is an isomorphism in $D^b_c(X, \overline{\mathbb{Q}}_l)$, namely*

$$K \simeq \bigoplus_i {}^{\mathfrak{p}}\mathcal{H}^i(K)[-i].$$

(2) *Let $P_0 \in \mathcal{P}_m(X_0, \overline{\mathbb{Q}}_l)$ be a pure perverse $\overline{\mathbb{Q}}_l$-sheaf on X_0. The pull-back P to X splits in $\mathcal{P}(X, \overline{\mathbb{Q}}_l)$ as a direct sum of intersection cohomology complexes associated with lisse irreducible sheaves on subvarieties of X.*

Remark 3.1.9. The splittings above do not necessarily hold over X_0.

If $K_0 \in D^b_c(X_0, \overline{\mathbb{Q}}_l)$, then the cohomology groups $H^*(X,K)$ on X are finite-dimensional $\overline{\mathbb{Q}}_l$-vector spaces with a continuous $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$-action and one can speak about the weights of $H^*(X,K)$, so that the notions of weights and purity extend to this context. In particular, this applies to the Ext-groups below.

We would like to give the reader a feeling of why weigths are related to splitting behaviours. These behaviors are governed by the Ext groups. Let $K_0, L_0 \in D^b_m(X_0, \overline{\mathbb{Q}}_l)$. The natural map $\mathrm{Ext}^1(K_0, L_0) \to \mathrm{Ext}^1(K,L)$ factors through the space of Frobenius invariants $\mathrm{Ext}^1(K,L)^{Fr}$, which is of pure of weight zero. If K_0 has weights $\leq w$ and L_0 has weights $\geq w'$, then $\mathrm{Ext}^1(K,L)$ has weights $\geq 1+w'-w$. If $w'=w$, then $\mathrm{Ext}^1(K,L)$ has weights ≥ 1, so that $\mathrm{Ext}^1(K,L)^{Fr}$ is trivial. The upshot is that given the right weights, a nontrivial extension class over $\mathbb{F}_q$ must become trivial over $\mathbb{F}$ and splittings may ensue (but only over the algebraic closure).

3.1.4. *The decomposition over* $\mathbb{F}$. With Theorem 3.1.8 in hand, it is immediate to prove the following theorem, which is one of the main results in [9].

Theorem 3.1.10 (Decomposition theorem and semisimplicity over $\mathbb{F}$). *Let $f_0 : X_0 \to Y_0$ be a proper morphism of $\mathbb{F}_q$-varieties, $K_0 \in D^b_c(X_0, \overline{\mathbb{Q}}_l)$ be pure and $f : X \to Y$ and K be the corresponding data over $\mathbb{F}$. There is an isomorphism in $D^b_c(Y, \overline{\mathbb{Q}}_l)$, namely*

$$f_*K \simeq \bigoplus_i {}^{\mathfrak{p}}\mathcal{H}^i(f_*K)[-i], \tag{17}$$

where each ${}^{\mathfrak{p}}\mathcal{H}^i(f_*K)$ splits as a direct sum of intersection cohomology complexes associated with lisse irreducible sheaves on subvarieties of Y. In particular, f_*K is semisimple; i.e., the unshifted summands ${}^{\mathfrak{p}}\mathcal{H}^i(K)$ are semisimple perverse sheaves on Y.

We now turn our attention to the relative hard Lefschetz theorem, also proved in [9]. Let $f_0 : X_0 \to Y_0$ be a morphism of $\mathbb{F}_q$-varieties, η_0 be the first Chern class of a line bundle η_0 on X_0. This defines a natural transformation $\eta_0 : f_{0*} \to f_{0*}[2](1)$. Here (1) is the Tate twist, lowering the weights by two; the reader unfamiliar with this notion may ignore the twist and still get a good idea of the meaning of the statements. By iterating, we obtain maps $\eta_0^i : f_{0*} \to f_{0*}[2i](i)$, $i \geq 0$. In particular, it defines natural transformations $\eta_0^i : {}^{\mathfrak{p}}\mathcal{H}^{-i}(f_{0*}(-)) \to {}^{\mathfrak{p}}\mathcal{H}^i(f_{0*}(-))(i)$.

Theorem 3.1.11 (Relative hard Lefschetz over $\mathbb{F}_q$ and $\mathbb{F}$). *Let P_0 be a pure perverse sheaf on X_0. Assume that X_0 is quasi-projective and that η_0 is a hyperplane bundle. Then the iterated cup product operation induces isomorphisms*

$$\eta_0^i : {}^{\mathfrak{p}}\mathcal{H}^{-i}(f_{0*}P_0) \xrightarrow{\simeq} {}^{\mathfrak{p}}\mathcal{H}^i(f_{0*}P_0)(i), \qquad \forall i \geq 0.$$

The same holds over $\mathbb{F}$ (with the understanding that P should come from a P_0).

Remark 3.1.12. The case $Y_0 = pt$, $P_0 = IC_{X_0}$, yields the hard Lefschetz theorem for intersection cohomology (over $\mathbb{F}_0$ and over $\mathbb{F}$). Using the same technique "from $\mathbb{F}$ to $\mathbb{C}$" in [9], §6, one sees that Theorem 3.1.11 implies the hard Lefschetz theorem for the intersection cohomology of complex projective varieties. An important precursor of the relative Hard Lefschetz theorem is P. Deligne's algebraic proof in [62] of the classical hard Lefschetz theorem.

3.1.5. *The decomposition theorem for complex varieties.* The technique "from $\mathbb{F}$ to $\mathbb{C}$" is used in [9], §6 to deduce the results of this section on complex algebraic varieties, from the results of the previous §3.1.4 on varieties defined over finite fields.

Let X be a complex variety. Consider the categories $\mathcal{D}_X$ of bounded constructible complexes of sheaves of *complex* vector spaces and its full subcategory of complex perverse sheaves $\mathcal{P}_X$. Recall that every perverse sheaf admits a finite filtration with simple quotients called the constituents of the perverse sheaf.

Definition 3.1.13 (**Perverse sheaves of geometric origin).** A perverse sheaf $P \in \mathcal{P}_X$ is said to be of *geometric origin* if it belongs to the smallest set such that (a) it contains the constant sheaf $\mathbb{C}_{pt}$ on a point, and such that it is stable under the following operations:
(b) for every map f, take the simple constituents of ${}^{\mathfrak{p}}\mathcal{H}^i(T(-))$, where $T = f^*, f_*, f_!, f^!$,
(c) take the simple constituents of ${}^{\mathfrak{p}}\mathcal{H}^i(- \otimes -)$, ${}^{\mathfrak{p}}\mathcal{H}^i(R\mathcal{H}om(-,-))$.

As a first example on a variety Z one may start with the map $g : Z \to pt$, take $g^*\mathbb{C}_{pt} = \mathbb{C}_Z$, and set P to be any simple constituent of one of the perverse complexes ${}^{\mathfrak{p}}\mathcal{H}^i(\mathbb{C}_Z)$. If $f : Z \to W$ is a map, one can take a simple constituent of ${}^{\mathfrak{p}}\mathcal{H}^j(f_*P)$ as an example on W. Another example consists of taking a simple local system of geometric origin L on a connected and smooth Zariski open subvariety $j : U \to X$ and setting $P := j_{!*}L[\dim U]$. Using either construction, we verify immediately that the intersection cohomology complex of a variety is of geometric origin.

Definition 3.1.14 (**Semisimple complexes of geometric origin).** A perverse sheaf P on X is said to be *semisimple of geometric origin* if it is a direct sum of simple perverse sheaves of geometric origin. A constructible complex $K \in \mathcal{D}_X$ is said to be *semisimple of geometric origin* if there is an isomorphism $K \simeq \bigoplus {}^{\mathfrak{p}}\mathcal{H}^i(K)[-i]$ in $\mathcal{D}_X$ and each perverse cohomology complex ${}^{\mathfrak{p}}\mathcal{H}^i(K)$ is semisimple of geometric origin.

We can now state the decomposition theorem and the relative hard Lefschetz theorems as they are stated and proved in [9]. If X is irreducible, then IC_X is simple of geometric origin so that the two theorems below apply to $K = IC_X$. The proofs can be found in [9], Théorèmes 6.2.5, 6.2.10. Note that while the results proved there are for sheaves of $\mathbb{C}$-vector spaces, one can deduce easily the variant for sheaves of $\mathbb{Q}$-vector spaces.

Theorem 3.1.15 (**Decomposition theorem over $\mathbb{C}$).** *Let $f : X \to Y$ be a proper morphism of complex varieties. If $K \in \mathcal{D}_X$ is semisimple of geometric origin, then so is f_*K.*

Theorem 3.1.16 (Relative hard Lefschetz theorem over $\mathbb{C}$). *Let $f : X \to Y$ be a projective morphism, P a perverse sheaf on X which is semisimple of geometric origin, η the first Chern class of an f-ample line bundle on X. Then the iterated cup product operation induces an isomorphism*

$$\eta^i \ : \ {}^{\mathfrak{p}}\mathcal{H}^{-i}(f_* P) \stackrel{\simeq}{\longrightarrow} {}^{\mathfrak{p}}\mathcal{H}^{i}(f_* P), \qquad \forall\, i \geq 0.$$

3.2. **M. Saito's approach via mixed Hodge modules.** The authors of [9] (cf. p.165) left open two questions: whether the decomposition theorem holds for the push-forward of the intersection cohomology complex of a local system underlying a polarizable variation of pure Hodge structures and whether it holds in the Kähler context. (Not all local systems as above are of geometric origin.)

In his remarkable work on the subject, M. Saito answered the first question in the affirmative in [156] and the second question in the affirmative in the case of IC_X in [158]; we refer the reader to M. Saito's paper for the precise formulations in the Kähler context. In fact, he developed in [157] a general theory of compatibility of mixed Hodge structures with the various functors, and in the process he completed the extension of the Hodge-Lefschetz theorems from the cohomology of projective manifolds to the intersection cohomology of projective varieties.

There are at least two important new ideas in his work. The former is that the Hodge filtration is to be obtained by a filtration at the level of D-modules. A precursor of this idea is Griffiths' filtration by the order of the pole. The latter is that the properties of his mixed Hodge modules are defined and tested using the vanishing cycle functor.

Saito's approach is deeply rooted in the theory of D-modules and, due to our ignorance on the subject, it will not be explained here. We refer to Saito's papers [156, 157, 158]. For a more detailed overview, see [32]. The papers [160] and [70] contain brief summaries of the results of the theory. See also [137].

Due to the importance of these results, we would like to discuss very informally Saito's achievements in the hope that even a very rough outline can be helpful to some. For simplicity only, we restrict ourselves to complex algebraic varieties (some results hold for complex analytic spaces).

Saito has constructed, for every variety Y, an abelian category $\mathrm{MHM}(Y)$ of mixed Hodge modules on Y. The construction is a tour-de-force which uses induction on dimension via a systematic use of the vanishing cycle functors associated with germs of holomorphic maps. It is in the derived category $D^b(\mathrm{MHM}(Y))$ that Saito's results on mixed Hodge structures can be stated and proved. If one is interested only in the decomposition and relative hard Lefschetz theorems, then it will suffice to work with the categories $MH(Y,w)$ below.

One starts with the abelian and semisimple category of polarizable Hodge modules of some weight $MH(Y,w)$. Philosophically they correspond to perverse pure complexes in $\overline{\mathbb{Q}}_l$-adic theory. Recall that, on a smooth variety, the Riemann-Hilbert correspondence, assigns to a regular holonomic D-module a perverse sheaf with complex coefficients. Roughly speaking, the simple objects are certain filtered regular holonomic D-modules $(\mathcal{M}, F)$. The D-module $\mathcal{M}$ corresponds, via an easy extension of the Riemann-Hilbert correspondence, to singular varieties, to the intersection cohomology complex of the complexification of a rational local system underlying a polarizable simple variation of pure Hodge structures of some weight (we omit the bookkeeping of weights).

Mixed Hodge modules correspond philosophically to perverse mixed complexes and are, roughly speaking, certain bifiltered regular holonomic D-modules $(\mathcal{M}, W, F)$ with the property that the graded objects $Gr_i^W \mathcal{M}$ are polarizable Hodge modules of weight i. The resulting abelian category MHM(Y) is not semisimple. However, the extensions are not arbitrary, as they are controlled by the vanishing cycle functor. The extended Riemann-Hilbert correspondence assigns to the pair $(\mathcal{M}, W)$ a filtered perverse sheaf (P, W) and this data extends to a functor of t-categories

$$\mathfrak{r} \;:\; D^b(\mathrm{MHM}(Y)) \longrightarrow \mathcal{D}_Y,$$

with the standard t-structure on $D^b(\mathrm{MHM}(Y))$ and the perverse t-structure on $\mathcal{D}_Y$. Beilinson's equivalence theorem [7], i.e., $\mathcal{D}_Y \simeq D^b(\mathcal{P}_Y)$, is used here, and in the rest of this theory, in an essential way. In fact, there is a second t-structure, say τ', on $D^b(\mathrm{MHM}(Y))$ corresponding to the standard one on $\mathcal{D}_Y$; see [157], Remarks 4.6.

The usual operations on D-modules induce a collection of operations on $D^b(\mathrm{MHM}(Y))$ that correspond to the usual operations on the categories $\mathcal{D}_Y$, i.e., $f^*, f_*, f_!, f^!$, tensor products, Hom, Verdier duality, nearby and vanishing cycle functors (cf. [157], Th. 0.1).

In the case when Y is a point, the category MHM(pt) is naturally equivalent to the category of graded polarizable rational mixed Hodge structures (cf. [157], p.319); here "graded" means that one has polarizations on the graded pieces of the weight filtration. At the end of the day, the W and F filtrations produce two filtrations on the cohomology and on the cohomology with compact supports of a complex in the image of $\mathfrak{r}$ and give rise to mixed Hodge structures compatible with the usual operations. The functor $\mathfrak{r}$ is exact and faithful, but not fully faithful (the map on Hom sets is injective, but not surjective), not even over a point: in fact, a pure Hodge structure of weight 1 and rank 2, e.g., H^1 of an elliptic curve, is irreducible as a Hodge structure, but not as a vector space.

The constant sheaf $\mathbb{Q}_Y$ is in the image of the functor $\mathfrak{r}$ and Saito's theory recovers Deligne's functorial mixed Hodge theory of complex varieties [59, 60]. See [157], p. 328 and [159], Corollary 4.3.

As mentioned above, mixed Hodge modules are a Hodge-theoretic analogue of the arithmetic mixed perverse sheaves discussed in §3.1. A mixed Hodge module $(\mathcal{M}, W, F) \in MHM(Y)$ is said to be pure of weight k if $Gr_i^W \mathcal{M} = 0$, for all $i \neq k$. In this case it is, by definition, a polarizable Hodge module so that a mixed Hodge module which is of some pure weight is analogous to an arithmetic pure perverse sheaf.

Saito proves the analogue of the arithmetic Corollary 3.1.5, i.e., if f is proper, then f_* preserves weights. Though the context and the details are vastly different, the rest of the story unfolds by analogy with the arithmetic case discussed in §3.1. A complex in $D^b(\mathrm{MHM}(Y))$ is said to be semisimple if it is a direct sum of shifted mixed Hodge modules which are simple and pure of some weight (= polarizable Hodge modules, i.e., associated with a simple variation of polarizable pure Hodge structures).

In what follows, note that the faithful functor $\mathfrak{r}$ commutes, up to natural equivalence, with the usual operations: e.g., $\mathfrak{r}(\mathcal{H}^j(M)) = {}^{\mathfrak{p}}\mathcal{H}^j(\mathfrak{r}(M))$, $f_*(\mathfrak{r}(M)) = \mathfrak{r}(f_*(M))$.

Theorem 3.2.1 (Decomposition theorem for polarizable Hodge modules). *Let $f : X \to Y$ be proper and $M \in D^b(MHM(X))$ be semisimple. Then the direct image $f_*M \in D^b(MHM(Y))$ is semisimple. More precisely, if $M \in MHM(X)$ is semisimple and pure, then*

$$f_*M \simeq \bigoplus_{j \in \mathbb{Z}} \mathcal{H}^j(f_*M)[-j],$$

where the $\mathcal{H}^j(f_*M) \in MHM(Y)$ are semisimple and pure.

Theorem 3.2.2 (Relative hard Lefschetz for polarizable Hodge modules). *Let $f : X \to Y$ be projective, $M \in MHM(X)$ be semisimple and pure and $\eta \in H^2(X, \mathbb{Q})$ be the first Chern class of an f-ample line bundle on X. Then the iterated cup product map is an isomorphism*

$$\eta^j : \mathcal{H}^{-j}(f_*M) \xrightarrow{\simeq} \mathcal{H}^j(f_*M)$$

of semisimple and pure mixed Hodge modules.

The proof relies on an inductive use, via Lefschetz pencils, of S. Zucker's [180] results on Hodge theory for degenerating coefficients in one variable.

The intersection cohomology complex of a polarizable variation of pure Hodge structures is the perverse sheaf associated with a pure mixed Hodge module (= polarizable Hodge module). This fact is not as automatic as in the case of the constant sheaf, for it requires the verification of the conditions of vanishing-cycle-functor-type involved in the definition of the category of polarizable Hodge modules. One may view this fact as the analogue of Gabber's purity theorem 3.1.6.

M. Saito thus establishes the decomposition and the relative hard Lefschetz theorems for coefficients in the intersection cohomology complex $IC_X(L)$ of a polarizable variation of pure Hodge structures, with the additional fact that one has mixed Hodge structures on the cohomology of the summands on Y and that the (noncanonical) splittings on the intersection cohomology group $IH(X, L)$ are compatible with the mixed Hodge structures of the summands. He has also established the hard Lefschetz theorem and the Hodge-Riemann bilinear relations for the intersection cohomology groups of projective varieties.

Saito's results complete the verification of the Hodge-Lefschetz package for the intersection cohomology groups of a variety Y, thus yielding the wanted generalization of the classical results in §1.1 to singular varieties.

The perverse and the standard truncations in $\mathcal{D}_Y$ correspond to the standard and to the above-mentioned τ' truncations in $D^b(\mathrm{MHM}(Y))$, respectively. See [157], p. 224 and Remarks 4.6. It follows that the following spectral sequences associated with complexes $K \in \mathfrak{r}(D^b(\mathrm{MHM}(Y))) \subseteq \mathcal{D}_Y$ are spectral sequences of mixed Hodge structures:

(1) the perverse spectral sequence;
(2) the Grothendieck spectral sequence;
(3) the perverse Leray spectral sequence associated with a map $f : X \to Y$;
(4) the Leray spectral sequence associated with a map $f : X \to Y$.

Remark 3.2.3. C. Sabbah [155] and T. Mochizuki [146] have extended the range of applicability of the decomposition theorem to the case of intersection cohomology complexes associated with semisimple local systems on quasi-projective varieties. They use, among other ideas, M. Saito's D-modules approach.

3.3. **A proof via classical Hodge theory.** Let us summarize some of our joint work on the subject of the decomposition theorem.

- Our paper [51] gives a geometric proof of the decomposition theorem for the push-forward f_*IC_X of the intersection cohomology complex via a proper map $f : X \to Y$ of complex algebraic varieties and complements it with a series of Hodge-theoretic results in the case when Y is projective. In particular, we endow the intersection cohomology groups of a projective variety with a pure Hodge structure. These results are stated in the case when X is nonsingular and projective as in Theorem 3.3.1 below. The statements in the case when X is projective, but possibly singular, are essentially identical to the ones in Theorem 3.3.1, except that one is required to replace $\mathbb{Q}_X[n]$ with IC_X (see [51]).
- In the paper [54], we show how to choose, when X and Y are projective, splitting isomorphisms in the decomposition theorem so that they are compatible with the various Hodge structures found in [51].
- The extension to the quasi-projective context of the results in [51, 54] is contained in [45], which builds on [55]. Since these papers deal with non-compact varieties, the statements involve mixed Hodge structures. These results are listed in §1.9.

Most of the results mentioned above have been obtained earlier and in greater generality by M. Saito in [156, 157] by the use of mixed Hodge modules. While our approach uses heavily the theory of perverse sheaves, it ultimately rests on classical and mixed Hodge theory.

The proof of the decomposition theorem in [51] is geometric in the sense that:

- it identifies the refined intersection forms on the fibers of the map f as the agent responsible for the splitting behavior of f_*IC_X and
- it provides a geometric interpretation of the perverse Leray filtration on $IH^*(X)$.

Since the mixed Hodge-theoretic results are surveyed in §1.9, in this section we mostly concentrate on outlining the proof of the decomposition theorem given in [51].

In the following two sections §3.3.1 and §3.3.2, we list the results contained in [51] and give an outline of the proofs in the key special case of a projective map $f : X \to Y$ of irreducible projective varieties with X *nonsingular* of dimension n.

We fix embeddings $X \subseteq \mathbb{P}$ and $Y \subseteq \mathbb{P}'$ into some projective spaces. We denote by $\mathbb{P}^\vee$ the projective space "dual" to $\mathbb{P}$, i.e., the projective space of hyperplanes in $\mathbb{P}$. Let η and L be the corresponding hyperplane line bundles on X and Y, respectively, and let $L' := f^*L$. We denote with the same symbol a line bundle, its first Chern class and the operation of cupping with it.

3.3.1. *The results when X is projective and nonsingular.* The following theorem summarizes some of the main results in [51] when X is projective and nonsingular. The results hold in the singular case as well, provided we replace $\mathbb{Q}_X[n]$ with IC_X. However, since the proof of the singular case relies on the proof of the nonsingular case, and this latter presents all the essential difficulties (see [52]), we prefer to discuss the nonsingular case only. Most of the results that follow hold in the case when X and Y are quasi-projective (see §1.9 and [45]). Recall that since X is nonsingular of dimension n, then $IC_X \simeq \mathbb{Q}_X[n]$.

Theorem 3.3.1. *Let $f : X \to Y$ be a proper map of projective varieties, with X nonsingular of dimension n. The following statements hold.*

(1) (**Decomposition theorem**) *$f_*\mathbb{Q}_X[n]$ splits as a direct sum of shifted intersection cohomology complexes with twisted coefficients on subvarieties of Y (cf.* §1.6.(7).(8)*).*

(2) (**Semisimplicity theorem**) *The summands are semisimple, i.e., the local systems* (8) *giving the twisted coefficients are semisimple. They are described below, following the refined intersection form theorem.*

(3) (**Relative hard Lefschetz theorem**)
Cupping with η yields isomorphisms

$$\eta^i : {}^{\mathfrak{p}}\mathcal{H}^{-i}(f_*\mathbb{Q}_X[n]) \simeq {}^{\mathfrak{p}}\mathcal{H}^{i}(f_*\mathbb{Q}_X[n]), \qquad \forall\, i \geq 0.$$

(4) (**Hodge structure theorem**) *The perverse t-structure yields the perverse filtration*

$$P^p H(X) = \operatorname{Im}\{H(Y, {}^{\mathfrak{p}}\tau_{\leq -p} f_*\mathbb{Q}_X[n]) \to H(Y, f_*\mathbb{Q}_X[n])\}$$

on the cohomology groups $H(X)$. This filtration is by Hodge substructures and the perverse cohomology groups

$$H^{a-n}(Y, {}^{\mathfrak{p}}\mathcal{H}^b(f_*\mathbb{Q}_X[n]) \simeq P^{-b}H^a(X)/P^{-b+1}H^a(X) = H^a_b(X),$$

i.e., the graded groups of the perverse filtration, inherit a pure Hodge structure.

(5) (**Hard Lefschetz theorems for perverse cohomology groups**) *The collection of perverse cohomology groups $H^*(Y, {}^{\mathfrak{p}}\mathcal{H}^*(f_*\mathbb{Q}_X[n])$ satisfies the conclusion of the hard Lefschetz theorem with respect to cupping with η on X and with respect to cupping with an L on Y, namely:*
The cup product $H^*(Y, {}^{\mathfrak{p}}\mathcal{H}^{-i}(f_*\mathbb{Q}_X[n])) \to H^{*+2i}(Y, {}^{\mathfrak{p}}\mathcal{H}^{i}(f_*\mathbb{Q}_X[n]))$ with η^i is an isomorphism for all $i \geq 0$.
The cup product with L^l: $H^{-l}(Y, {}^{\mathfrak{p}}\mathcal{H}^{i}(f_*\mathbb{Q}_X[n])) \to H^{l}(Y, {}^{\mathfrak{p}}\mathcal{H}^{i}(f_*\mathbb{Q}_X[n]))$ is an isomorphism for all $l \geq 0$ and all i.

(6) (**The perverse filtration on $H^*(X)$**) *The perverse filtration on the groups $H^r(X)$ is given by the following equation (where it is understood that a linear map with a nonpositive exponent is defined to be the identity and that kernels and images are inside of $H^r(X)$):*

$$P^p H^r(X) \;=\; \sum_{a+b=n-(p+r)} \operatorname{Ker} L'^{a+1} \cap \operatorname{Im} L'^{-b}.$$

(7) (**Generalized Lefschetz decomposition and Hodge-Riemann bilinear relations**) *Let $i, j \in \mathbb{Z}$ and consider the perverse cohomology groups of (4). Define $P^{-j}_{-i} := \operatorname{Ker}\eta^{i+1} \cap \operatorname{Ker} L^{j+1} \subseteq H^{n-i-j}_{-i}(X)$ if $i, j \geq 0$ and $P^{-j}_{-i} := 0$, otherwise. There is a Lefschetz-type direct sum decomposition (the (η, L)-decomposition) into pure Hodge substructures*

$$H^{n-i-j}_{-i}(X) = \bigoplus_{l,m\,\in\mathbb{Z}} \eta^{-i+l} L^{-j+m} P^{j-2m}_{i-2l}.$$

Define, for $i, j \geq 0$, bilinear forms on $H^{n-i-j}_{-j}(X)$:

$$S^{\eta L}_{ij}(\alpha, \beta) := \int_X \eta^i \wedge L^j \wedge \alpha \wedge \beta.$$

These forms are well defined and, using the hard Lefschetz theorems (5), they can be suitably defined for every $i, j \in \mathbb{Z}$. *The bilinear forms* $S^{\eta,L}_{ij}$ *are nondegenerate and orthogonal with respect to the* (η, L)*-decomposition. Up to the sign* $(-1)^{i+j-m-l+1}$, *these forms are a polarization (see* §5.2, *especially* (38)) *of each* (η, L)*-direct summand.*

(8) (**Generalized Grauert contractibility criterion**) *Fix* $y \in Y$ *and* $j \in \mathbb{Z}$. *The natural class map, obtained by composing the push-forward in homology with Poincaré duality*

$$H_{n-j}(f^{-1}(y)) \longrightarrow H^{n+j}(X),$$

is naturally filtered in view of the decomposition theorem. The resulting graded class map

$$H_{n-j,j}(f^{-1}(y)) \longrightarrow H^{n+j}_{j}(X)$$

is an injection of pure Hodge structures polarized in view of the generalized Hodge-Riemann relations (7).

(9) (**Refined intersection form theorem**) *The refined intersection form*

$$H_{n-j}(f^{-1}(y)) \longrightarrow H^{n+j}(f^{-1}(y))$$

(see §5.4, *Refined intersection forms) is naturally filtered in view of the decomposition theorem, and the resulting graded refined intersection form*

$$H_{n-j,k}(f^{-1}(y)) \longrightarrow H^{n+j}_{k}(f^{-1}(y))$$

is zero for $j \neq k$ *and an isomorphism for* $j = k$.

3.3.2. *An outline of the proof of Theorem* 3.3.1. We start by sketching the proof in the nontrivial toy model of a semismall map ([50]), as many important steps appear already in this case. We refer to §4.2 for basic definitions and facts concerning this remarkable class of maps.

1. The case of semismall maps.

There is no loss of generality in assuming that the map f is surjective. Since a semismall map is generically finite, we have $n = \dim X = \dim Y$. We proceed by induction on $n = \dim Y$ and prove all the results of Theorem 3.3.1.

By the semismallness assumption, we have that ${}^{\mathfrak{p}}\mathcal{H}^j(f_*\mathbb{Q}_X[n]) = 0$ for every $j \neq 0$, so that the relative hard Lefschetz theorem is trivial and so is the perverse filtration. The first point to show is that, *from the point of view of the Hodge-Lefschetz package,* $L' = f^*L$ *behaves as if it were a hyperplane line bundle,* even though it is not (it is trivial along the fibers of the map f): *all the theorems in* §5.2.1 *hold with* L' *replacing* η.

The hard Lefschetz theorem for L'. By induction, we assume that the statements in Theorem 3.3.1 hold for all semismall maps between varieties of dimension less than n. Let $D \subseteq Y$ be a generic hyperplane section. The map $f^{-1}(D) \to D$ is still semismall. Since $f_*\mathbb{Q}_X[n]$ is perverse, in the range $i \geq 2$ the Lefschetz theorem on hyperplane sections for perverse sheaves (see §2.6) reduces the hard Lefschetz theorem for L'^i on X to that for L'^{i-1} on $f^{-1}(D)$. In the critical case $i = 1$, the cup product with L' factors as $H^{n-1}(X) \to H^{n-1}(f^{-1}(D)) \to H^{n+1}(X)$, where the first map is injective and the second is surjective. As explained in the "inductive approach to hard Lefschetz" paragraph of §5.2, the inductive Hodge-Riemann relations for the restriction of L' to $f^{-1}(D)$ give the hard Lefschetz theorem for the cup product with L'.

The approximation trick. We must prove the Hodge-Riemann relations for the space of primitives $P^n_{L'} = \operatorname{Ker} L' : H^n(X) \to H^{n+2}(X)$ (for use in the case when $\dim X = n+1$). The hard Lefschetz theorem discussed above implies that $\dim P^n_{L'} = b_n - b_{n-2}$ and that the decomposition $H^n(X) = P^n_{L'} \oplus L' H^{n-2}(X)$ is orthogonal with respect to the Poincaré pairing, just as if L' were a hyperplane bundle. In particular, the restriction of the Poincaré pairing $S(\alpha, \beta) = \int_X \alpha \wedge \beta$ to $P^n_{L'}$ is nondegenerate. The bilinear form $\widetilde{S}(\alpha, \beta) := S(\alpha, C\beta)$ (C is the Weil operator; see §5.2) is still nondegenerate. The class L' is on the boundary of the ample cone: for any positive integer r, the class $L' + \frac{1}{r}\eta$ is ample, and we have the classical Hodge-Riemann relations on the subspace $P^n_{L_r} := \operatorname{Ker}(L' + \frac{1}{r}\eta) \subseteq H^n(X)$: the remark made above on the dimension of $P^n_{L'}$ implies that any class $\alpha \in P^n_{L'}$ is the limit of classes $\alpha_r \in P^n_{L_r}$, so that the restriction of $\widetilde{S}$ to $P^n_{L'}$ is semidefinite; since it is also nondegenerate, the Hodge-Riemann bilinear relations follow.

Decomposition and semisimplicity. To prove the decomposition and semisimplicity theorems, we proceed one stratum at a time; higher-dimensional strata are dealt with inductively by cutting transversally with a generic hyperplane section D on Y, so that one is reduced to the semismall map $f^{-1}(D) \to D$, where the dimension of a positive-dimensional stratum on Y has decreased by one unit on D. The really significant case left is that of a zero-dimensional relevant stratum S. As explained in §4.2.1, the splitting of the perverse sheaf $f_*\mathbb{Q}_X[n]$ into a direct sum of intersection cohomology complexes with twisted coefficients on subvarieties of Y is equivalent to the nondegeneracy of the refined intersection form $I : H_n(f^{-1}(y)) \times H_n(f^{-1}(y)) \longrightarrow \mathbb{Q}$, for $y \in S$.

In order to establish the nondegeneracy of the refined intersection forms I, we turn to mixed Hodge theory (§5.2) and use the following result of P. Deligne (cf. [60], Proposition 8.2.6):

(Weight miracle). *If $Z \subseteq U \subseteq X$ are inclusions with X a nonsingular compact variety, $U \subseteq X$ a Zariski dense open subvariety and $Z \subseteq U$ a closed subvariety of X, then the images in $H^j(Z, \mathbb{Q})$ of the restriction maps from X and from U coincide.*

Thanks to the weight miracle, $H_n(f^{-1}(y))$ injects in $H^n(X)$ as a Hodge substructure. Since, for a general section D, we have $f^{-1}(y) \cap f^{-1}(D) = \emptyset$, we see that $H_n(f^{-1}(y))$ is contained in $P^n_{L'}$. The restriction of the Poincaré pairing to $H_n(f^{-1}(y))$ is thus a polarization and is hence nondegenerate. The same is thus true for the refined intersection form I.

As noted already in Theorem 4.2.7, the local systems involved have finite monodromy; hence they are obviously semisimple. This concludes our discussion of the semismall case.

2. The general case: extracting the semismall "soul" of a map.

The proof is by induction on the pair of indices $(\dim Y, r(f))$, where $r(f) = \dim X \times_Y X - \dim X$ is the *defect of semismallness* of the map f. To give an idea of the role played by $r(f)$, let us say that in the decomposition theorem §1.6(7), the direct sum ranges precisely in the interval $[-r(f), r(f)]$. The inductive hypothesis takes the following form: all the statements of Theorem 3.3.1 hold for all proper maps $g : X' \to Y'$ with either $r(g) < r(f)$, or with $r(g) = r(f)$ and $\dim Y' < \dim Y$. Let $n := \dim X$. The induction starts with the verification of Theorem 3.3.1 in the case when Y is a point, in which case the results boil down to the classical result of Hodge-Lefschetz theory outlined in §1.1 and listed more succinctly in Theorem 5.2.1.

2a. The universal hyperplane section and relative hard Lefschetz theorem.

Let $g : X' \subseteq X \times \mathbb{P}^\vee \to Y' = Y \times \mathbb{P}^\vee$ be the universal hyperplane section. If $r(f) > 0$, then $r(g) < r(f)$ and, by induction, Theorem 3.3.1 holds for g. As in the classical case (cf. §5.2.1), the relative Lefschetz hyperplane Theorem 2.6.4 implies the relative hard Lefschetz theorem for f except for $i = 1$, where we have the factorization of the cup product map with η:

$$ {}^{\mathfrak{p}}\mathcal{H}^{-1}(f_*\mathbb{Q}_X[n]) \xrightarrow{\rho} {}^{\mathfrak{p}}\mathcal{H}^{0}(g_*\mathbb{Q}_{X'}[n-1]) \xrightarrow{\gamma} {}^{\mathfrak{p}}\mathcal{H}^{1}(f_*\mathbb{Q}_X[n]). $$

The first map is a monomorphism and the second is an epimorphism. We argue as in the proof of the hard Lefschetz theorem via the semisimplicity of monodromy: we use an argument similar to the identification of the monodromy invariants of a Lefschetz pencil with the image of the cohomology of a variety into the cohomology of a hyperplane section, and we couple it with the semisimplicity (inductive assumption!) of ${}^{\mathfrak{p}}\mathcal{H}^{0}(g_*\mathbb{Q}_{X'}[n-1])$ to show that:

Proposition 3.3.2. *The image of* ${}^{\mathfrak{p}}\mathcal{H}^{-1}(f_*\mathbb{Q}_X[n])$ *in* ${}^{\mathfrak{p}}\mathcal{H}^{0}(g_*\mathbb{Q}_{X'}[n+1])$ *is a split summand applied isomorphically onto* ${}^{\mathfrak{p}}\mathcal{H}^{1}(f_*\mathbb{Q}_X[n])$ *by* γ*.*

The relative hard Lefschetz theorem for f follows and, by applying Deligne's Lefschetz splitting criterion, Theorem 5.3.1, we conclude that $f_*\mathbb{Q}_X[n] \simeq \bigoplus_i {}^{\mathfrak{p}}\mathcal{H}^{i}(f_*\mathbb{Q}_X[n])[-i]$.

From the statements known for g by induction, we get that ${}^{\mathfrak{p}}\mathcal{H}^{i}(f_*\mathbb{Q}_X[n])$ is a direct sum of intersection cohomology complexes of semisimple local systems for all $i \neq 0$. Moreover, for all $i \neq 0$, the associated perverse cohomology groups verify the hard Lefschetz theorem and the Hodge-Riemann relations with respect to cupping with L.

What is left to investigate is the zero perversity complex ${}^{\mathfrak{p}}\mathcal{H}^{0}(f_*\mathbb{Q}_X[n])$. Again in analogy with the classical case, we can "shave off" another piece which comes from the hyperplane section and dispose of it by using the inductive hypothesis. In fact, the analogue of the primitive Lefschetz decomposition theorem 5.2.1(2) holds: by setting, for every $i \geq 0$, $\mathcal{P}^{-i} := \operatorname{Ker}\{\eta^{i+1} : {}^{\mathfrak{p}}\mathcal{H}^{-i}(f_*\mathbb{Q}_X[n]) \to {}^{\mathfrak{p}}\mathcal{H}^{i+2}(f_*\mathbb{Q}_X[n])\}$ we have canonical direct sum decompositions:

$$ (18) \qquad {}^{\mathfrak{p}}\mathcal{H}^{-i}(f_*\mathbb{Q}_X[n]) = \bigoplus_{r\geq 0} \eta^r \mathcal{P}^{-i-2r}, \qquad {}^{\mathfrak{p}}\mathcal{H}^{i}(f_*\mathbb{Q}_X[n]) = \bigoplus_{r\geq 0} \eta^{i+r} \mathcal{P}^{-i-2r}. $$

The only remaining pieces for which we have to prove the statements of Theorem 3.3.1 are the perverse sheaf $\mathcal{P}^0$ and its cohomology $H^*(Y, \mathcal{P}^0)$, which, in view of the primitive decomposition, is a summand of the perverse cohomology group $H_0^{*+n}(X)$. (The analogy with the classical study of algebraic varieties by means of hyperplane sections is as follows: the new cohomology classes, i.e., the ones not coming from a hyperplane section, appear only in the middle dimension $P^n = \operatorname{Ker}\{\eta : H^n(X) \to H^{n+2}(X)\}$. In this game, "middle dimension" is recentered at zero.) We are left with proving:

(1) The Hodge package of §5.2.1 holds for $H^*(Y, \mathcal{P}^0)$ with respect to cupping with L.
(2) $\mathcal{P}^0$ is a direct sum of twisted intersection cohomology complexes.
(3) The twisting local systems are semisimple.

2b. The Hodge package for $\mathcal{P}^0$.

The main intuition behind the proof of the statements (1) and (2) above, which was inspired also by the illuminating discussion of the decomposition theorem contained in [133], is that $H^*(Y, \mathcal{P}^0)$ is the "semismall soul of the map f"; that is, *it behaves as the cohomology of a (virtual) nonsingular projective variety with a semismall map to Y*. In order to handle the group $H^*(Y, \mathcal{P}^0)$, we mimic the proof of the decomposition theorem for semismall maps.

One of the main difficulties in [51] is that, in order to use classical Hodge theory, we have to prove at the outset that the perverse Leray filtration is Hodge-theoretic, i.e., that the subspaces $P^pH^*(X) \subseteq H^*(X)$ (cf. §3.3.1(4)) are Hodge substructures of the natural Hodge structure on $H^*(X)$. The geometric description of the perverse filtration in [55] (see §2.4) implies that this fact holds for every algebraic map, proper or not, to a quasi-projective variety, and the proof in [55] is independent of the decomposition theorem. It follows that the geometric description of the perverse filtration in [55] can therefore be used to yield a considerable simplification of the line of reasoning in [51] for it endows, at the outset, the perverse cohomology groups $H^a_b(X)$ with a natural Hodge structure, compatible with the primitive Lefschetz decompositions stemming from (18), and with respect to which the cup product maps $L : H^*(Y, \mathcal{P}^i) \to H^{*+2}(Y, \mathcal{P}^i)$ and $\eta : P^kH^*(X) \to P^{k-2}H^{*+2}(X)$ are Hodge maps of type $(1, 1)$.

We start by proving (1), i.e., the Hodge package for $H^*(Y, \mathcal{P}^0)$. The argument for the hard Lefschetz isomorphism $L^i : H^{-i}(Y, \mathcal{P}^0) \simeq H^i(Y, \mathcal{P}^0)$ is completely analogous to the one used for a semismall map: the Lefschetz theorem on hyperplane sections for the perverse sheaf $\mathcal{P}^0$ and the inductive hypothesis (for a generic hyperplane section $D \subseteq Y$, we have $f' : f^{-1}(D) \to D$ and $\mathcal{P}^0$ restricts, up to a shift, to the analogous complex $\mathcal{P}'^0$ for f') yield immediately the theorem in the range $i \geq 2$ and also yield a factorization of $L : H^{-1}(Y, \mathcal{P}^0) \to H^1(Y, \mathcal{P}^0)$ as the composition of the injective restriction to D and the surjective Gysin map. Again by the inductive hypotheses, the Poincaré pairing polarizes $\operatorname{Ker} L : H^0(D, \mathcal{P}'^0) \to H^2(D, \mathcal{P}'^0)$, and, as in the classical case, this proves the remaining case $i = 1$.

The most delicate point is to prove that the Riemann-Hodge relations hold for $P^{00} := \operatorname{Ker}\{L : H^0(Y, \mathcal{P}^0) \to H^2(Y, \mathcal{P}^0)\}$. The Poincaré pairing induces a bilinear form S on $H^n(X) = H^0(f_*\mathbb{Q}_X[n])$ and on its subquotient $H^0(Y, \mathcal{P}^0)$. This is because we have the following orthogonality relation: $P^1H(X) \subseteq P^0H(X)^\perp$. More is true: S is nondegenerate on $P^0H^n(X)/P^1H^n(X) = H^n_0(X)$ and the (η, L)-decomposition is orthogonal so that the restriction of S to the summand P^{00} is nondegenerate. The Hodge-Riemann relations are then proved with an "approximation trick" similar, although more involved, to the one used in the semismall case. We consider the subspace $\Lambda = \lim_r \operatorname{Ker}(L' + \frac{1}{r}\eta) \subseteq H^n(X)$. Clearly, we have $\Lambda \subseteq \operatorname{Ker} L'$ and the hard Lefschetz theorem implies that $\operatorname{Ker} L' \subseteq P^0H^n(X)$. The nondegenerate form $\widetilde{S}$ is semidefinite on $\Lambda/\Lambda \cap P^1H^n(X)$. It follows that it is a a polarization. A polarization restricted to a Hodge substructure is still a polarization. The Hodge-Riemann relations for P^{00} follow from the inclusion of Hodge structures $P^{00} \subseteq \Lambda/\Lambda \cap P^1H^n(X)$.

2c. Semisimplicity.

We need to prove that $\mathcal{P}^0$ splits as a direct sum of intersection cohomology complexes of semisimple local systems. As in the case of semismall maps, higher-dimensional strata are disposed of by induction on the dimension of Y and by cutting with generic hyperplane sections of Y. One is left to prove the critical case of a zero-dimensional stratum. Again by the splitting criterion of Remark 5.7.5, we have to prove that, for any point y in the zero-dimensional stratum, denoting by $i : y \to Y$ the closed imbedding, $\iota : \mathcal{H}^0(i^!\mathcal{P}^0) \to \mathcal{H}^0(i^*\mathcal{P}^0)$ is an isomorphism. Given the decomposition (18), $\mathcal{H}^0(i^!\mathcal{P}^0)$ is a direct summand of $H_n(f^{-1}(0))$ and $\mathcal{H}^0(i^*\mathcal{P}^0)$ is a direct summand of $H^n(f^{-1}(0))$, so that the map ι is the restriction to these summands of the refined intersection form (§5.4) on $f^{-1}(0)$. Although in general, the map $H_n(f^{-1}(0)) \to H^n(X)$ is not injective, the weight miracle is used to prove that the map $\mathcal{H}^0(i^!\mathcal{P}^0) \to H^n_0(X)$ is an injection with image a pure Hodge substructure of the Hodge structure we have on $H^n_0(X)$ (by virtue of the geometric description of the perverse filtration [55] mentioned above). Since this image lands automatically in the L'-primitive part, we conclude that the descended intersection form polarizes this image; hence ι is an isomorphism and we have the desired splitting into intersection cohomology complexes.

We still have to establish the semisimplicity of the local systems in (8) (and hence of the ones appearing in $\mathcal{P}^0$). This is accomplished by exhibiting them as quotients of local systems associated with smooth proper maps and are hence semisimple by the semisimplicity for smooth proper maps, Theorem 5.2.2.

This concludes the outline of the proof of Theorem 3.3.1.

4. Applications of perverse sheaves and of the decomposition theorem

In this section, we give, without any pretense of completeness, a sample of remarkable applications of the theory of perverse sheaves and of the decomposition theorem.

We focus mostly on the complex case, although most of the discussion goes through over a field of positive characteristic, with constructible $\mathbb{Q}$-sheaves replaced by l-adic ones.

In this chapter, we use the machinery of derived categories and functors and some results on perverse sheaves. The notions introduced in our crash course may not be sufficient to follow the (few) proofs included. We refer to §5.3, to the references quoted there, and to §2. In particular, we adopt the simplified notation $f_*, f_!$ for the derived functors $Rf_*, Rf_!$.

4.1. **Toric varieties and combinatorics of polytopes.** The purpose of this section is to state and explain the content of Theorem 4.1.6 on how the combinatorics of rational polytopes in Euclidean space relates to the intersection cohomology groups of the associated toric varieties. Theorem 4.1.6 is stated in §4.1.1 and we work out two examples in §4.1.2, where the decomposition theorem is seen in action in situations where, we hope, the minimal background we provide in this section is sufficient to follow the arguments.

For the basic definitions concerning toric varieties, we refer to [77, 154]. The recent survey [23] contains many historical details, motivation, a discussion of open problems and recent results, and an extensive bibliography.

We will adopt the point of view of polytopes, which we find more appealing to intuition.

Recall that a d-dimensional normal projective complex variety X is a toric variety if it has an action of the complex torus $T = (\mathbb{C}^*)^d$ with finitely many orbits. In this case, there is a moment map $\mu : X \to \mathbb{R}^d$ whose image is a d-dimensional convex polyhedron P, whose vertices have rational coordinates, and which determines the toric variety X up to isomorphism. The mapping μ determines an order-preserving one-to-one correspondence between the orbits of T and the faces of P as follows. For each orbit $\mathcal{O} \subseteq X$ the image $\mu(\mathcal{O}) \subseteq P$ is the interior F^0 of a face $F \subseteq P$. Moreover, $\dim_{\mathbb{C}}(\mathcal{O}) = \dim_{\mathbb{R}}(F)$ and the fibers of $\mu : \mathcal{O} \to F^0$ are diffeomorphic to the compact torus $(S^1)^{\dim F}$. For $i = 0, \ldots, d-1$, let f_i be the number of i-dimensional faces of P. We denote by X_P the projective toric variety associated with P. A d-dimensional simplex Σ_d is the convex envelope of $d+1$ affinely independent points $v_0, \ldots, v_d$ in $\mathbb{R}^d$. X_{Σ_d} is a possibly weighted d-dimensional projective space. A polytope is said to be simplicial if its faces are simplices. We say that a toric variety is $\mathbb{Q}$-smooth when it has only finite quotient singularities. A map of varieties $f : \widetilde{X} \to X$, both of which are toric, is called a toric resolution if it is birational, equivariant with respect to the torus action, and $\widetilde{X}$ is $\mathbb{Q}$-smooth.

The following is well known:

Proposition 4.1.1. *A toric variety X_P is $\mathbb{Q}$-smooth if and only if P is simplicial.*

4.1.1. *The h-polynomial.* Let P be a simplicial d-dimensional polytope with number of faces encoded by the “face vector” $(f_0, \ldots, f_{d-1})$. Define the associated “h-polynomial”

$$h(P,t) = (t-1)^d + f_0(t-1)^{d-1} + \ldots + f_{d-1}. \tag{19}$$

The simplicial toric variety X_P has a decomposition as a disjoint union of locally closed subsets, each isomorphic to the quotient of an affine space by a finite commutative group. This decomposition can be used to compute the rational cohomology groups $H^*(X_P, \mathbb{Q})$, and we have the following proposition; see [77], Section 5.2 for a detailed proof.

Proposition 4.1.2. *Let P be a simplicial rational polytope, with “h-polynomial” $h(P,t) = \sum_0^d h_k(P)t^k$. Then*

$$H^{2k+1}(X_P, \mathbb{Q}) = 0 \quad \text{and} \quad \dim H^{2k}(X_P, \mathbb{Q}) = h_k(P).$$

Poincaré duality and the hard Lefschetz theorem imply the following.

Corollary 4.1.3. *We have the following relations:*

$$h_k(P) = h_{d-k}(P) \quad \text{for } 0 \leq k \leq d, \qquad h_{k-1}(P) \leq h_k(P) \quad \text{for } 0 \leq k \leq \lfloor d/2 \rfloor.$$

Corollary 4.1.3 amounts to a set of nontrivial relations among the face numbers f_i and gives necessary conditions for a sequence $(a_0, \ldots, a_{d-1}) \in \mathbb{N}^d$ to be the face vector of a simplicial polytope. Exploiting more fully the content of the hard Lefschetz theorem, it is possible to characterize completely the sequences in $\mathbb{N}^d$ which occur as the face vectors of some simplicial polytope; see [23], Theorem 1.1.

The polynomial

$$g(P,t) = h_0 + (h_1 - h_0)t + \ldots + (h_{[d/2]} - h_{[d/2]-1})t^{[d/2]} \tag{20}$$

has, by Corollary 4.1.3, positive coefficients and uniquely determines h. The coefficient $g_l = h_l - h_{l-1}$ is the dimension of the primitive cohomology (§5.2) of X_P in degree l.

Example 4.1.4. Let Σ_d be the d-dimensional simplex. We have $f_0 = d+1 = \binom{d+1}{1}, \ldots, f_i = \binom{d+1}{i+1}$ and

$$\begin{aligned} h(\Sigma_d, t) &= (t-1)^d + \binom{d+1}{1} + \ldots + \binom{d+1}{i+1}(t-1)^{d-i-1} + \ldots + \binom{d+1}{d} \\ &= 1 + t + \ldots + t^d, \end{aligned}$$

so that $h_i = 1$ and $g(\Sigma_d, t) = 1$, consistently with the fact that $X_{\Sigma_d} = \mathbb{P}^d$.

Let C_2 be the square, convex envelope of the four points $(\pm 1, 0), (0, \pm 1)$. We have $f_0 = 4, f_1 = 4$, $h(C_2, t) = (t-1)^2 + 4(t-1) + 4 = t^2 + 2t + 1$, and $g(C_2, t) = 1 + t$. In fact, $X_{C_2} = \mathbb{P}^1 \times \mathbb{P}^1$.

Similarly, for the octahedron O_3, the convex envelope of $(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)$, we have $f_0 = 6, f_1 = 12, f_2 = 8$, $h(O_3, t) = t^3 + 3t^2 + 3t + 1$ and $g(O_3, t) = 2t + 1$. This is in accordance with the Betti numbers of $X_{O_3} = (\mathbb{P}^1)^3$.

If the polytope is not simplicial, so that the toric variety is not $\mathbb{Q}$-smooth, neither Poincaré duality nor the hard Lefschetz theorem necessarily hold for the cohomology groups. Furthermore, as shown in [138], the ordinary cohomology group of a singular toric variety is not a purely combinatorial invariant, but depends also on some geometric data of the polytope, e.g., the measures of the angles between the faces of the polytope. The situation drastically simplifies when considering intersection cohomology groups. In fact, Poincaré duality and the hard Lefschetz theorem hold for intersection cohomology, so that the "generalized" h-polynomial $h(P, t) = \sum_0^d h_k(P) t^k$, where $h_k(P) := \dim IH^{2k}(X_P, \mathbb{Q})$, satisfies the conclusions of Corollary 4.1.3. Furthermore, it turns out that the polynomial $h(P, t)$ is a combinatorial invariant, i.e., it can be defined only in terms of the partially ordered set of faces of the polytope P. Note that when the polytope P is simplicial, so that the toric variety X_P is $\mathbb{Q}$-smooth, then $H^*(X_P, \mathbb{Q}) = IH^*(X_P, \mathbb{Q})$. Hence, in this case, by Proposition 4.1.2, the generalized h-polynomial defined below coincides with the one defined earlier and we can denote the two in the same way.

We now give the combinatorial definitions of the h and g polynomials for a not necessarily simplicial polytope.

Definition 4.1.5. Suppose P is a polytope of dimension d and that the polynomials $g(Q, t)$ and $h(P, t)$ have been defined for all convex polytopes Q of dimension less than d. We set

$$h(P, t) = \sum_{F < P} g(F, t)(t-1)^{d-1-\dim F},$$

where the sum is extended to all proper faces F of P including the empty face $\emptyset$, for which $g(\emptyset, t) = h(\emptyset, t) = 1$ and $\dim \emptyset = -1$. The polynomial $g(P, t)$ is defined from $h(P, t)$ as in (20).

We note that these definitions coincide with the previous ones given in (19) and (20) if P is simplicial, since $g(\Sigma, t) = 1$; see Example 4.1.4. In fact, we have the following.

Theorem 4.1.6 ([74])**.** *Let P be a rational polytope. Then*

$$h(P, t) = \sum_{F < P} g(F, t)(t-1)^{d-1-\mathrm{dim F}} = \sum \dim IH^{2k}(X_P, \mathbb{Q}) t^k.$$

Given a subdivision $\widetilde{P}$ of the polytope P, there is a corresponding map $X_{\widetilde{P}} \to X_P$. The toric orbits of X_P provide a stratification for f. The fibers over toric orbits, which properties can be read off from the combinatorics of the subdivision, are unions of toric varieties glued along toric subvarieties; for a discussion, see [101]. It is well known (cf. [77], Section 2.6) that any polytope becomes simplicial after a sequence of subdivisions.

Theorem 4.1.6 on the dimension of the intersection cohomology groups of a toric variety can be proved by exploiting the decomposition theorem for a resolution defined by a subdivision of the polytope P. A sketch of a proof along these lines has been given by R. MacPherson in several talks in 1982. J. Bernstein and A. Khovanskii also developed proofs, which have not been published.

4.1.2. *Two worked-out examples of toric resolutions.* We describe MacPherson's approach to Theorem 4.1.6 via the decomposition theorem in the special cases of subdivision of the cube of dimensions 3 and 4. The general case can be proved along these lines.

Let C_i be the i-dimensional cube. It is not simplicial if $i > 2$, and the k-dimensional faces of C_i are k-dimensional cubes C_k. The three-dimensional cube C_3 has 8 faces of dimension 0 and 12 faces of dimension 1, which are of course simplicial; there are 6 faces of dimension 2, for which we have already computed $g(C_2, t) = 1 + t$. It follows that

(21)
$$h(C_3, t) = (t-1)^3+8(t-1)^2+12(t-1)+6(1+t) = 1+5t+5t^2+t^3 \text{ and } g(C_3, t) = 1+4t.$$

Similarly, the four-dimensional cube C_4 has 16 faces of dimension 0, 32 faces of dimension 1, which are all simplicial, 24 faces of dimension 2, which are equal to C_2, and finally 8 faces of dimension 3, which are equal to C_3. Thus

$$\begin{aligned} h(C_4, t) &= (t-1)^4 + 16(t-1)^3 + 32(t-1)^2 + 24(1+t)(t-1) + 8(1+4t) \\ &= t^4 + 12t^3 + 14t^2 + 12t + 1. \end{aligned} \tag{22}$$

The 3-dimensional cube C_3 has a simplicial subdivision C_3' which does not add any vertex and divides every two-dimensional face into two simplices by adding its diagonal; see the picture in [77], p.50. The resulting map $f : X_{C_3'} \to X_{C_3}$ is an isomorphism outside the six singular points of X_{C_3}, and the fibers over these points are isomorphic to $\mathbb{P}^1$. The f-vector of C_3' has $f_0 = 8$, $f_1 = 18$ and $f_2 = 12$ and h-polynomial $h(C_3', t) = t^3 + 5t^2 + 5t + 1$ which equals the h-polynomial $h(C_3, t)$ computed above. This equality reflects the fact that f is a small resolution in the sense of Remark 4.2.4 below, so that $H^i(X_{C_3'}) = IH^i(X_{C_3})$.

We discuss the decomposition theorem for the map $f : X_{\widetilde{C_3}} \to X_{C_3}$, where $\widetilde{C_3}$ is obtained by the following decomposition of C_3: for each of the six two-dimensional faces F_i, we add its barycenter P_{F_i} as a new vertex, and we join P_{F_i} with each vertex of F_i. We obtain in this way a simplicial polytope $\widetilde{C_3}$ with 14 vertices, 36 edges and 24 two-dimensional simplices. Its h-polynomial is $h(\widetilde{C_3}, t) = t^3+11t^2+11t+1$. The map f is an isomorphism away from the six points $p_1, \ldots, p_6$ corresponding to the two-dimensional faces of C_3. The fibers D_i over each point p_i comprise the toric variety corresponding to C_2, i.e., $\mathbb{P}^1 \times \mathbb{P}^1$; in particular, $H^4(D_i) = \mathbb{Q}$ and

${}^{\mathfrak{p}}\mathcal{H}^{\pm 1}(f_*\mathbb{Q}_{X_{\widetilde{C_3}}}[3]) \simeq \bigoplus \mathbb{Q}_{p_i}$. The decomposition theorem for f reads as follows:

$$f_*\mathbb{Q}_{X_{\widetilde{C_3}}}[3] \;\simeq\; IC_{C_3} \,\oplus\, (\bigoplus_i \mathbb{Q}_{p_i}[1]) \,\oplus\, (\bigoplus_i \mathbb{Q}_{p_i}[-1])$$

and

$$H^l(X_{\widetilde{C_3}}) \simeq IH^l(X_{C_3}) \text{ for } l \neq 2,4,$$
$$\dim H^l(X_{\widetilde{C_3}}) = \dim IH^l(X_{C_3}) + 6 \text{ for } l = 2,4.$$

It follows that $\sum \dim IH^{2k}(X_{C_3})t^k = \sum \dim H^{2k}(X_{\widetilde{C_3}})t^k - 6t - 6t^2 = h(\widetilde{C_3}, t) - 6t - 6t^2 = t^3 + 5t^2 + 5t + 1 = h(C_3, t)$, as already computed.

Finally, as a more challenging example, we consider the four-dimensional cube C_4. We subdivide it by adding as new vertices the barycenters of the 8 three-dimensional faces and of the 24 two-dimensional faces. It is not hard to see that the f-vector of the resulting simplicial polytope $\widetilde{C_4}$ is $(48, 240, 384, 192)$ and $h(\widetilde{C_4}, t) = t^4 + 44t^3 + 102t^2 + 44t + 1$. The geometry of the map $f : X_{\widetilde{C_4}} \to X_{C_4}$ which is relevant to the decomposition theorem is the following. The 24 two-dimensional faces correspond to rational curves $\overline{O}_i$, closures of one-dimensional orbits O_i, along which the map f is locally trivial and looks, on a normal slice, just as the map $X_{\widetilde{C_3}} \to X_{C_3}$ examined in the example above. The fiber over each of the 8 points p_i corresponding to the three-dimensional faces is isomorphic to $X_{\widetilde{C_3}}$. Each point p_i is the intersection of the six rational curves $\overline{O}_{i_j}$ corresponding to the six faces of the three-dimensional cube associated with p_i. The last crucial piece of information is that *the local systems arising in the decomposition theorem are in fact trivial*. Roughly speaking, this follows from the fact that the fibers of the map f along a fixed orbit depend only on the combinatorics of the subdivision of the corresponding face. We thus have ${}^{\mathfrak{p}}\mathcal{H}^{\pm 1}(f_*\mathbb{Q}_{X_{\widetilde{C_4}}}[4])_{|O_i} \simeq \bigoplus_i \mathbb{Q}_{O_i}[1]$ and ${}^{\mathfrak{p}}\mathcal{H}^{\pm 2}(f_*\mathbb{Q}_{X_{\widetilde{C_4}}}[4]) \simeq \bigoplus_i H^6(f^{-1}(p_i)) \simeq \bigoplus_i H^6(\widetilde{C_3})_{p_i} \simeq \bigoplus_i \mathbb{Q}_{p_i}$. The decomposition theorem reads:

$$f_*\mathbb{Q}_{X_{\widetilde{C_4}}}[4] \simeq IC_{C_4} \,\oplus\, (\bigoplus_i V_{p_i}) \,\oplus\, (\bigoplus_i (IC_{\overline{O}_i}[1] \oplus IC_{\overline{O}_i}[-1]))$$
$$\oplus\, (\bigoplus_i (\mathbb{Q}_{p_i}[2] \oplus \mathbb{Q}_{p_i}[-2])).$$

The vector spaces V_{p_i} are subspaces of $H^4(f^{-1}(p_i))$ and contribute to the zero perversity term ${}^{\mathfrak{p}}\mathcal{H}^0(f_*\mathbb{Q}_{X_{\widetilde{C_4}}}[4])$. In order to determine their dimension, we compute the stalk

$$\mathcal{H}^0(f_*\mathbb{Q}_{X_{\widetilde{C_4}}}[4])_{p_i} = H^4(f^{-1}(p_i)) = H^4(\widetilde{C_3}).$$

As we saw above, $\dim H^4(\widetilde{C_3}) = 11$. By the support condition $\mathcal{H}^0(\mathcal{I}C_{C_4}) = 0$ and, since $\mathcal{I}C_{\overline{O}_i} = \mathbb{Q}_{\overline{O}_i}[1]$, we get

$$11 = \dim \mathcal{H}^0(f_*\mathbb{Q}_{X_{\widetilde{C_4}}}[4])_{p_i} = \dim V_{p_i} \oplus (\bigoplus_{\overline{O}_j \ni p_i} \mathcal{H}^{-1}(\mathcal{I}C_{\overline{O}_j})) = \dim V_{p_i} + 6,$$

since only six curves $\overline{O}_j$ pass through p_i. Hence $\dim V_{p_i} = 5$ and finally

$$f_*\mathbb{Q}_{X_{\widetilde{C_4}}}[4] \simeq \mathcal{I}C_{C_4} \oplus (\bigoplus_{i=1}^{8} (\mathbb{Q}_{p_i}^{\oplus 5} \oplus \mathbb{Q}_{p_i}[2] \oplus \mathbb{Q}_{p_i}[-2]) \oplus (\bigoplus_{i=1}^{24} (\mathbb{Q}_{\overline{O}_i} \oplus \mathbb{Q}_{\overline{O}_i}[2])).$$

By taking the cohomology we get:

$$\begin{aligned}\sum \dim IH^{2k}(X_{C_4})t^k &= \sum \dim H^{2k}(X_{\widetilde{C_4}})t^k - 8(t+5t^2+t^3) - 24(t+2t^2+t^3)\\ &= t^4+44t^3+102t^2+44t+1-8(t+5t^2+t^3)-24(t+2t^2+t^3)\\ &= t^4+12t^3+14t^2+12t+1 = h(C_4,t),\end{aligned}$$

as computed in (22).

4.2. **Semismall maps.** Semismall maps occupy a very special place in the applications of the theory of perverse sheaves to geometric representation theory. Surprisingly, many maps which arise naturally from Lie-theoretic objects are semismall. In a sense which we will try to illustrate in the discussion of the examples below, the semismallness of a map is related to the semisimplicity of the algebraic object under consideration. We limit ourselves to proper and surjective semismall maps with a nonsingular domain.

In the case of semismall maps, the decomposition theorem takes the particularly simple form of Theorem 4.2.7. Corollary 4.2.8, on the semisimplicity of the algebra of endomorphisms of the direct image, is a simple consequence.

As we have shown in [48], the proof of Theorem 4.2.7 is reduced to the proof of the nondegeneration of certain bilinear forms defined on the homology groups of the fibers via intersection theory. We discuss this point of view in §4.2.1.

We discuss two examples of semismall maps: the resolution of the nilpotent cone (§4.2.2) and the resolution of the n-th symmetric product of a nonsingular surface via a Hilbert scheme of n points on the surface (§4.2.3). In the first case, the decomposition theorem leads to a simplified description of the Springer correspondence; this correspondence (see Theorem 4.2.14) gives a geometric realization of the Weyl group of a semisimple linear algebraic group and its representations. In the second case, we recall the basic geometric facts about Hilbert schemes that led to the remarkably explicit Theorem 4.2.16.

A *stratification* for f is a decomposition of Y into finitely many locally closed nonsingular subsets such that $f^{-1}(S_k) \to S_k$ is a topologically locally trivial fibration. The subsets S_k are called *strata*.

The following easy observation makes perverse sheaves enter this picture.

Proposition 4.2.1. *Let X be a connected nonsingular n-dimensional variety, and $f : X \to Y$ be a proper surjective map of varieties. Let $Y = \coprod_{k=0}^{n} S_k$ be a stratification for f. Let $y_k \in S_k$ and set $d_k := \dim f^{-1}(y_k) = \dim f^{-1}(S_k) - \dim S_k$. The following are equivalent:*

1. *$f_*\mathbb{Q}_X[n]$ is a perverse sheaf on Y;*
2. *$\dim X \times_Y X \leq n$;*
3. *$\dim S_k + 2d_k \leq \dim X$, for every $k = 0, \ldots, n$.*

Definition 4.2.2. A proper and surjective map f satisfying one of the equivalent properties in Proposition 4.2.1 is said to be *semismall*.

Definition 4.2.3. Let X, Y, S_k and d_k be as in Proposition 4.2.1. A stratum S_k is said to be *relevant* if $\dim S_k + 2d_k = \dim X$.

A semismall map $f : X \to Y$ must be finite over an open dense stratum in Y in view of property (3). Hence, semismall maps are generically finite. The converse is not true, e.g., the blowing up of a point in $\mathbb{C}^3$. Note that, since $\dim Y = \dim X$, a relevant stratum has even codimension.

Remark 4.2.4. If the stronger inequalities $\dim S_k + 2d_k < \dim X$ are required to hold for every nondense stratum, then the map is said to be *small*. In this case, $f_* \mathbb{Q}_X[n]$ satisfies the support and co-support conditions for intersection cohomology ((12),(13) of §2.1). Hence, if $Y_o \subseteq Y$ denotes a nonsingular dense open subset over which f is a covering, then we have that $f_* \mathbb{Q}_X[n] = IC_Y(L)$, where L is the local system $f_* \mathbb{Q}_{X_{|Y_o}}$.

Example 4.2.5. Surjective maps between surfaces are always semismall. A surjective map of threefolds is semismall iff no divisor $D \subseteq X$ is contracted to a point on Y.

A great wealth of examples of semismall maps is furnished by contractions of (holomorphic) symplectic varieties, which we now describe. A nonsingular quasi-projective complex variety is called *holomorphic symplectic* if there is a holomorphic 2-form $\omega \in \Gamma(X, \Omega^2_X)$ which is closed and nondegenerate; that is, $d\omega = 0$ and $\omega^{\frac{\dim X}{2}}$ does not vanish at any point. The following is proved in [109]:

Theorem 4.2.6. *Let X be a quasi-projective holomorphic symplectic variety, and $f : X \to Y$ a projective birational map. Then f is semismall.*

Some important examples of semismall maps which are contractions of holomorphic symplectic varieties will be considered in §4.2.2 and §4.2.3.

The decomposition theorem for a semismall map takes a particularly simple form: the only contributions come from the relevant strata S and they consist of nontrivial summands $IC_{\overline{S}}(L)$, where the local systems L turn out to have finite monodromy.

Let S be a relevant stratum, $y \in S$ and let $E_1, \ldots, E_l$ be the irreducible $\dim S$-dimensional components of $f^{-1}(y)$. The monodromy of the E_i's defines a group homomorphism ρ_S of the fundamental group $\pi_1(S, y)$ to the group of permutations of the E_i's, and, correspondingly, a local system of $\mathbb{Q}$-vector spaces L_S. The semisimplicity of L_S then follows immediately from the fact that the monodromy factors through a finite group. With this notation, let us give the statement of the decomposition theorem in the case of semismall maps:

Theorem 4.2.7 (Decomposition theorem for semismall maps). *Let I_{rel} be the set of relevant strata, and, for each $S \in I_{rel}$, let L_S be the corresponding local system with finite monodromy defined above. There is a canonical isomorphism in $\mathcal{P}_Y$:*

$$f_* \mathbb{Q}_X[n] \simeq \bigoplus_{S \in I_{rel}} IC_{\overline{S}}(L_S). \tag{23}$$

Let $Irr(\pi_1(S))$ be the set of irreducible representations of $\pi_1(S, y)$. For $\chi \in Irr(\pi_1(S))$, we denote by $\mathcal{L}_\chi$ the corresponding local system on S. We have an isotypical decomposition in the category $\pi_1(S)$-Mod of representations of $\pi_1(S)$:

$$\rho_S \simeq \bigoplus_{\chi \in Irr(\pi_1(S))} \chi \otimes V_S^\chi,$$

where V_S^χ is a vector space whose dimension is the multiplicity of the representation χ in ρ_S. Correspondingly, we have a decomposition of local systems $L_S = \bigoplus_{\chi \in Irr(\pi_1(S))} \mathcal{L}_\chi \otimes V_S^\chi$, and, for each term $IC_{\overline{S}}(L_S)$ in (23), an isotypical decomposition

$$IC_{\overline{S}}(L_S) \simeq \bigoplus_{\chi \in Irr(\pi_1(S))} IC_{\overline{S}}(\mathcal{L}_\chi) \otimes V_S^\chi. \tag{24}$$

The second special feature of semismall maps concerns the endomorphism algebra $\mathrm{End}_{\mathcal{D}_Y}(f_*\mathbb{Q}_X[n])$; see [39, 41] for details.

By Schur's lemma, for χ and $\mathcal{L}_\chi$ as above, we have that

$$\mathrm{End}_{\mathcal{D}_Y}(IC(\mathcal{L}_\chi)) = \mathrm{End}_{\mathcal{D}_S}(\mathcal{L}_\chi) = \mathrm{End}_{\pi_1(S)-\mathrm{Mod}}(\chi)$$

is a division ring R_χ. The intersection cohomology sheaves $IC(\mathcal{L}_\chi)$ are simple objects in the category of perverse sheaves and Theorem 4.2.7 can be restated by saying that $f_*\mathbb{Q}_X[n]$ is a semisimple perverse sheaf. We thus have the following.

Corollary 4.2.8 (**Semisimplicity of the endomorphism algebra**). *Let $f : X \to Y$ be a semismall map. Then the endomorphism algebra $\mathrm{End}_{\mathcal{D}_Y}(f_*\mathbb{Q}_X[n])$ is semisimple, that is, isomorphic to a direct sum of matrix algebras over division rings. In fact, we have:*

$$\mathrm{End}_{\mathcal{D}_Y}(f_*\mathbb{Q}_X[n]) \simeq \bigoplus_{S \in I_{rel}} \mathrm{End}_{\mathcal{D}_Y}(IC_{\overline{S}}(L_S)) \simeq \bigoplus_{\substack{S \in I_{rel} \\ \chi \in Irr(\pi_1(S))}} R_\chi \otimes \mathrm{End}(V_S^\chi). \tag{25}$$

Furthermore, if $H_{2n}^{BM}(X \times_Y X)$ is given the structure of an algebra coming from the composition of correspondences, then there is an isomorphism of algebras (see [41], Lemma 2.23)

$$\mathrm{End}_{\mathcal{D}_Y}(f_*\mathbb{Q}_X[n]) \simeq H_{2n}^{BM}(X \times_Y X). \tag{26}$$

The endomorphism algebra contains in particular the idempotents giving the projection of $f_*\mathbb{Q}_X[n]$ on the irreducible summand of the canonical decomposition (23). Since, again by semismallness, $H_{2n}^{BM}(X \times_Y X)$ is the top-dimensional Borel Moore homology, it is generated by the irreducible components of $X \times_Y X$. The projectors are therefore realized by algebraic correspondences.

This has been pursued in [50], where we prove, in accordance with the general philosophy of [41], a "motivic" refinement of the decomposition theorem in the case of semismall maps. In particular, it is possible to construct a (relative) Chow motive corresponding to the intersection cohomology groups of singular varieties which admit a semismall resolution.

4.2.1. *Semismall maps and intersection forms.* Let $f : X \to Y$ be a semismall map. Every stratum yields a bilinear form on a certain homology group which has a neat geometric interpretation in terms of basic intersection theory on X. Theorem 4.2.9 below states that the decomposition theorem for the semismall map f turns out to be equivalent to the nondegeneracy of all these intersection forms.

Let us describe these intersection forms. If a stratum is not relevant, then, as noted below, the construction that follows yields a trivial homology group. Let $S \subseteq Y$ be a relevant stratum, and $y \in S$. Let Σ be a local transversal slice to S at y, given for example by intersecting a small contractible Euclidean neighborhood of y with the complete intersection of $\dim S$ general hyperplane sections in Y passing through y. The restriction $f_| : f^{-1}(\Sigma) \to \Sigma$ is still semismall and $d = \dim f^{-1}(y) = (1/2)\dim f^{-1}(\Sigma)$. By composing the chain of maps:

$$H_{2d}(f^{-1}(y)) = H_{2d}^{BM}(f^{-1}(y)) \longrightarrow H_{2d}^{BM}(f^{-1}(\Sigma)) \simeq H^{2d}(f^{-1}(\Sigma)) \longrightarrow H^{2d}(f^{-1}(y)),$$

where the first map is the push-forward with respect to a closed inclusion and the second is the restriction, we obtain the *intersection pairing* (cf. §5.4) *associated with the relevant stratum S*,

$$I_S : H_{2d}(f^{-1}(y)) \times H_{2d}(f^{-1}(y)) \longrightarrow \mathbb{Q}.$$

Of course, we have used the usual identification $\mathrm{Bil}(U,U) \simeq \mathrm{Hom}(U,U^\vee)$. If the stratum is not relevant, then $\dim f^{-1}(y) < d$ and $H_{2d}(f^{-1}(y)) = 0$, and the intersection form is defined, it is trivial and also nondegenerate, in the sense that the corresponding linear map is an isomorphism of trivial vector spaces.

A basis of $H_{2d}(f^{-1}(y))$ is given by the classes of the d-dimensional irreducible components $E_1, \ldots, E_l$ of $f^{-1}(y)$. The intersection pairing I_S is then represented by the intersection matrix $||E_i \cdot E_j||$ of these components, computed in the, possibly disconnected, manifold $f^{-1}(\Sigma)$.

In what follows, for simplicity only, let us assume that $S = S_k$ is a connected stratum of dimension k, relevant or not. Let $U = \coprod_{k'>k} S_{k'}$ be the union of the strata of dimension strictly greater than k and $U' = U \amalg S$. Denote by $i : S \to U' \longleftarrow U : j$ the corresponding imbeddings. The intersection map $H_{2d}(f^{-1}(y)) \to H^{2d}(f^{-1}(y))$ is then identified with the natural map of stalks

$$\mathcal{H}^{-d}(i^! f_* \mathbb{Q}_{U'}[n])_y \longrightarrow \mathcal{H}^{-d}(i^* f_* \mathbb{Q}_{U'}[n])_y.$$

By Remark 5.7.5, the nondegeneracy of I_S is equivalent to the existence of a canonical isomorphism:

$$f_* \mathbb{Q}_{U'}[n] \simeq j_{!*} f_* \mathbb{Q}_{U'}[n] \oplus \mathcal{H}^{-\dim S}(i^! f_* \mathbb{Q}_{U'}[n])[\dim S]. \tag{27}$$

It follows that the splitting behaviour of $f_* \mathbb{Q}_X[n]$ is governed precisely by the nondegeneracy of the forms I_S.

In our paper, cf. [48], we proved, using classical Hodge-Lefschetz theory, that for every relevant stratum S with typical fiber of dimension d, the form I_S has a precise sign. In particular, all forms I_S are nondegenerate. We summarize these results in the following.

Theorem 4.2.9. *Let $f : X \to Y$ be a semismall map with X nonsingular. Then the statement of the decomposition theorem is equivalent to the nondegeneracy of the intersection forms I_S. These forms are nondegenerate and if a connected component of a stratum S is relevant with typical fiber of dimension d, then the form $(-1)^d I_S$ is positive definite.*

4.2.2. *Examples of semismall maps I: Springer theory.* References for what follows are [39, 167]. Let G be a semisimple connected linear algebraic group with Lie algebra $\mathfrak{g}$, let $T \subseteq G$ be a maximal torus, let B be a Borel subgroup containing T and let W be the Weyl group. The *flag variety* G/B is complete and parametrizes the Borel subalgebras of $\mathfrak{g}$. We recall that an element $x \in \mathfrak{g}$ is *nilpotent* (resp. *semisimple*) if the endomorphism $[x, -] : \mathfrak{g} \to \mathfrak{g}$ is nilpotent (resp. diagonalizable). If $\dim \mathrm{Ker}\,[x, -\,]$ equals the dimension of T, then x is said to be *regular*.

Let $\mathcal{N} \subseteq \mathfrak{g}$ be the cone of nilpotent elements of $\mathfrak{g}$. It can be easily shown (cf. [39]) that

$$\widetilde{\mathcal{N}} = \{(x, \mathfrak{c}) \in \mathcal{N} \times G/B : \mathfrak{c} \text{ is a Borel subalgebra of } \mathfrak{g} \text{ and } x \in \mathcal{N} \cap \mathfrak{c}\}$$

is isomorphic to the cotangent bundle $T^* G/B$ of the flag variety G/B, and is therefore endowed with a natural (exact) holomorphic symplectic form. The map

$p : \widetilde{\mathcal{N}} \to \mathcal{N} \subseteq \mathfrak{g}$, defined as $p(x, \mathfrak{c}) = x$, is surjective, since every nilpotent element is contained in a Borel subalgebra, generically one-to-one, since a generic nilpotent element is contained in exactly one Borel subalgebra, proper, since G/B is complete, and semismall, since $\widetilde{\mathcal{N}}$ is holomorphic symplectic. The map p is called the Springer resolution.

Example 4.2.10. If $G = SL_2$, then the flag variety $G/B = \mathbb{P}^1$ and the cotangent space is the total space of the line bundle $\mathcal{O}_{\mathbb{P}^1}(-2)$. The variety obtained by contracting the zero-section to a point is isomorphic to the cone with equation $z^2 = xy$ in $\mathbb{C}^3$. If $\{H, X, Y\}$ denotes the usual basis of $\mathfrak{sl}_2$, the matrix $zH + xX - yY$ is nilpotent precisely when $z^2 = xy$.

The aim of the *Springer correspondence* is to get an algebra isomorphism between the rational group algebra of the Weyl group W of $\mathfrak{g}$ and the algebra of correspondences of $\widetilde{\mathcal{N}}$,

$$\mathbb{Q}[W] \xrightarrow{\simeq} H^{BM}_{2\dim \widetilde{\mathcal{N}}}(\widetilde{\mathcal{N}} \times_{\mathcal{N}} \widetilde{\mathcal{N}}) ,$$

so that the elements of the Weyl group will correspond to certain correspondences in the fiber product above.

The Springer correspondence is realized as follows. One constructs an action of the Weyl group W on $p_*\mathbb{Q}_{\widetilde{\mathcal{N}}}[\dim \widetilde{\mathcal{N}}]$. This action extends to an algebra homomorphism $\mathbb{Q}[W] \to \mathrm{End}_{\mathcal{D}_{\mathcal{N}}}(p_*\mathbb{Q}_{\widetilde{\mathcal{N}}}[\dim \widetilde{\mathcal{N}}])$ which is verified to be an isomorphism. Finally, one uses (26).

We now sketch, following [126] (see also [16, 17]), the construction of the desired W-action. By a theorem of Chevalley, there is a map $q : \mathfrak{g} \to \mathfrak{t}/W$ defined as follows: any $x \in \mathfrak{g}$ has a unique expression $x = x_{ss} + x_n$, where x_{ss} is semisimple and x_n is nilpotent and commutes with x_{ss}. Then x_{ss} is conjugate to an element of $\mathfrak{t}$, well defined up to the action of W. The quotient $\mathfrak{t}/W$ is an affine space. Let us denote by $\mathfrak{t}^{rs} = \mathfrak{t} \setminus \{\text{root hyperplanes}\}$, the set of regular elements in $\mathfrak{t}$, and by $\mathfrak{g}^{rs} = q^{-1}(\mathfrak{t}^{rs}/W)$ the set of regular semisimple elements in $\mathfrak{g}$. The set $\mathfrak{t}^{rs}/W$ is the complement of a divisor $\Delta \subseteq \mathfrak{t}/W$. The map $q : \mathfrak{g}^{rs} \to \mathfrak{t}^{rs}/W$ is a fibration with fiber G/T and we have the monodromy representation $\rho : \pi_1(\mathfrak{t}^{rs}/W) \to \mathrm{Aut}(H^*(G/T))$.

Example 4.2.11. Let $G = SL_n$. The map q sends a traceless matrix to the coefficients of its characteristic polynomial. The set $\mathfrak{t}^{rs}/W = \mathfrak{t}/W \setminus \Delta$ is the set of polynomials with distinct roots. The statement that the map $q : \mathfrak{g}^{rs} \to \mathfrak{t}^{rs}/W$ is a fibration boils down to the fact that a matrix commuting with a diagonal matrix with distinct eigenvalues must be diagonal and that the adjoint orbit of such a matrix is closed in $\mathfrak{sl}_n$.

Let us define

$$\widetilde{\mathfrak{g}} = \{(x, \mathfrak{c}) \in \mathfrak{g} \times G/B : \mathfrak{c} \text{ is a Borel subalgebra of } \mathfrak{g} \text{ and } x \in \mathfrak{c}\}.$$

Let $p : \widetilde{\mathfrak{g}} \to \mathfrak{g}$ be the projection to the first factor. This map "contains" the Springer resolution in the sense that $\widetilde{\mathcal{N}} = p^{-1}(\mathcal{N}) \subseteq \widetilde{\mathfrak{g}}$.

On the other hand, the G-orbits in $\widetilde{\mathfrak{g}}$ of regular semisimple elements (i.e., for which the corresponding x is regular semisimple) are affine varieties isomorphic to G/T and diffeomorphic to $\widetilde{\mathcal{N}}$.

Example 4.2.12. In the case discussed in Example 4.2.10, one considers the family of affine quadrics $Y_t \subseteq \mathbb{C}^3$ of the equation $z^2 = xy + t$ for $t \in \mathbb{C}$. For $t \neq 0$, Y_t is diffeomorphic, but not isomorphic, to $T^*\mathbb{P}^1$, while, for $t = 0$, Y_0 is the nilpotent cone of $\mathfrak{sl}_2$. Pulling back this family by the map $t \to t^2$, we get the family $z^2 = xy + t^2$, which admits a simultaneous (small) resolution, whose fiber at $t = 0$ is the map $T^*\mathbb{P}^1 \to Y_0$.

The Weyl group acts simply transitively on the set of Borel subgroups containing a regular semisimple element. Setting $\widetilde{\mathfrak{g}}^{rs} = p^{-1}(\mathfrak{g}^{rs})$, this observation leads to the following:

Proposition 4.2.13. *The restriction $p' : \widetilde{\mathfrak{g}}^{rs} \to \mathfrak{g}^{rs}$ is an (unramified) covering map with Galois group W. The map $p : \widetilde{\mathfrak{g}} \to \mathfrak{g}$ is small.*

We summarize what we have discussed so far in the following diagram (the map r is defined below):

$$\begin{tikzcd}
& & & \widetilde{\mathcal{N}} \arrow[dl, hook'] \arrow[dd, "p"] & \\
\widetilde{\mathfrak{g}}^{rs} \arrow[rr, hook] \arrow[dd, "p'"'] & & \widetilde{\mathfrak{g}} \arrow[rr] \arrow[dd, "p"] & & \mathfrak{t} \arrow[dd] \\
& G/T \arrow[rr, "r"] \arrow[dl, hook'] & & \mathcal{N} \arrow[dl, hook', "i"] & \\
\widetilde{\mathfrak{g}}^{rs}/W = \mathfrak{g}^{rs} \arrow[rr, hook] & & \mathfrak{g} \arrow[rr, "q"] & & \mathfrak{t}/W
\end{tikzcd}$$

Let $L = p'_*\mathbb{Q}_{\widetilde{\mathfrak{g}}^{rs}}$ be the local system associated with the W-covering, which, by its very definition, is endowed with an action of the Weyl group W. Since any map between local systems extends uniquely to a map between the associated intersection cohomology complexes (see [87], Theorem 3.5), we have an action of W on $IC_{\mathfrak{g}}(L)$. Since p is small, by Remark 4.2.4, $IC_{\mathfrak{g}}(L) = p_*\mathbb{Q}_{\widetilde{\mathfrak{g}}}[\dim \mathfrak{g}]$.

In particular, there is an action of W on $i^*p_*\mathbb{Q}_{\widetilde{\mathfrak{g}}}[\dim\mathfrak{g}] = p_*\mathbb{Q}_{\widetilde{\mathcal{N}}}[\dim\widetilde{\mathcal{N}}]$, and this is the sought-for W-action.

A perhaps more intuitive way to realize this action is the following. We have $\mathcal{N} = q^{-1}(0)$. There is a continuous retraction map $r : G/T \to \mathcal{N}$. Since the affine variety G/T is diffeomorphic to $\widetilde{\mathcal{N}}$, we have an isomorphism:

$$r_*\mathbb{Q}_{G/T}[\dim\widetilde{\mathcal{N}}] \simeq p_*\mathbb{Q}_{\widetilde{\mathcal{N}}}[\dim\widetilde{\mathcal{N}}].$$

As we have already observed, the monodromy of the fibration $q : \mathfrak{g}^{rs} \to \mathfrak{t}^{rs}/W$ gives an action of $\pi_1(\mathfrak{t}^{rs}/W)$ on $r_*\mathbb{Q}_{G/T}[\dim\widetilde{\mathcal{N}}]$. There is an exact sequence of groups:

$$0 \to \pi_1(\mathfrak{t}^{rs}) \to \pi_1(\mathfrak{t}^{rs}/W) \to W \to 0,$$

and the existence of the simultaneous resolution $\widetilde{\mathfrak{g}}$ shows that the monodromy factors through an action of W, and this yields the desired alternative description of the W-action on p_* etc.

As mentioned earlier, the W-action extends to an algebra homomorphism

$$\mathbb{Q}[W] \longrightarrow \mathrm{End}_{\mathcal{D}_{\mathcal{N}}}(p_*\mathbb{Q}_{\widetilde{\mathcal{N}}}[\dim\widetilde{\mathcal{N}}]) \;\; = \;\; H^{BM}_{2\dim\widetilde{\mathcal{N}}}(\widetilde{\mathcal{N}} \times_{\mathcal{N}} \widetilde{\mathcal{N}})$$

and we have

Theorem 4.2.14 ([16])**.** *The map*

$$\mathbb{Q}[W] \longrightarrow H^{BM}_{2\dim \widetilde{\mathcal{N}}}(\widetilde{\mathcal{N}} \times_{\mathcal{N}} \widetilde{\mathcal{N}}) \tag{28}$$

constructed above is an isomorphism of algebras.

We thus have a geometric construction of the group ring of the Weyl group W as an algebra of (relative) correspondences on $\widetilde{\mathcal{N}}$, and a natural basis given by the irreducible components of $\widetilde{\mathcal{N}} \times_{\mathcal{N}} \widetilde{\mathcal{N}}$.

A deeper investigation of the isomorphism (28) sheds light on the irreducible representations of $\mathbb{Q}[W]$, or, equivalently, of W, by giving a natural geometric construction of these representations: the nilpotent cone $\mathcal{N}$ has a natural G-invariant stratification, given by the orbits of the adjoint action contained in $\mathcal{N}$, i.e., by the conjugacy classes of nilpotent elements. Let $\mathrm{Conj}(\mathcal{N})$ be the set of conjugacy classes of nilpotent elements in $\mathfrak{g}$. For $[x] \in \mathrm{Conj}(\mathcal{N})$, let x be a representative, and denote by $\mathfrak{B}_x := p^{-1}(x)$ the fiber over x and by $S_x = Gx$ the stratum of $\mathcal{N}$ containing x.

Example 4.2.15. Let $G = SL_n$. Each conjugacy class contains exactly one matrix which is a sum of Jordan matrices, so that the G-orbits are parameterized by the partitions of the integer n. The open dense stratum of $\mathcal{N}$ corresponds to the Jordan block of length n.

It can be proved (cf. [169, 164]) that every stratum S_x is relevant and that all the components of $\mathfrak{B}_x$ have the same dimension d_x. The vector space $H_{2d_x}(\mathfrak{B}_x)$, generated by the irreducible components of $\mathfrak{B}_x$, is, by construction, a representation of W. This representation is not necessarily irreducible, as the finite group of connected components of the stabilizer G_x of x acts. This action commutes with the action of W and splits $H_{2d_x}(\mathfrak{B}_x)$. It can be shown that every irreducible representation of W is realized as a direct summand of some $H_{2d_x}(\mathfrak{B}_x)$. At this point, we refer the reader to the original papers [165, 16, 17], and to the book [39].

4.2.3. *Examples of semismall maps II: Hilbert schemes of points.* A reference for what follows is [149]. The n-th symmetric product $(\mathbb{C}^2)^{(n)} = (\mathbb{C}^2)^n/\mathcal{S}_n$, parametrizing 0-cycles $Z = \sum_k n_k p_k$ of $\mathbb{C}^2$ of length n, is singular. Singularities appear when some points come together, that is, at cycles $\sum_k n_k p_k$ where some multiplicity is greater than one. The Hilbert scheme $X = (\mathbb{C}^2)^{[n]}$ is a certain resolution of singularities of n-th $(\mathbb{C}^2)^{(n)} = (\mathbb{C}^2)^n/\mathcal{S}_n$ which keeps tracks of the "tangent" information when two or more points collapse. For instance, $X = (\mathbb{C}^2)^{[2]}$ is the blowup of $X = (\mathbb{C}^2)^{(2)}$ along the diagonal, consisting of cycles of type $2p_1$.

When n points come together at a point p_0 of coordinates (x_0, y_0), this tangent information is encoded as a *scheme* structure supported on the point parametrizing the cycle np_0. This scheme structure is given in terms of an ideal I of the ring of polynomials in two indeterminates $\mathbb{C}[X,Y]$ with radical $\sqrt{I} = (X - x_0, Y - y_0)$, the maximal ideal of the point p_0, such that $\dim_{\mathbb{C}}\mathbb{C}[X,Y]/I = n$. In general, the points of the variety $X = (\mathbb{C}^2)^{[n]}$ parametrize ideals $I \subseteq \mathbb{C}[X,Y]$ such that $\dim_{\mathbb{C}}\mathbb{C}[X,Y]/I = n$. Every such ideal is the product $\prod I_k$ of ideals supported at points $p_k \in \mathbb{C}^2$, and we can associate with it the 0-cycle $Z(I) := \sum_k n_k p_k$, where $n_k = \dim_{\mathbb{C}}\mathbb{C}[X,Y]/I_k$, called the support of this ideal. Then $n = \sum_k n_k$ and $Z(I)$ is a point in the symmetric product $(\mathbb{C}^2)^{(n)} = (\mathbb{C}^2)^n/\mathcal{S}_n$. The *Hilbert-Chow map* $\pi : (\mathbb{C}^2)^{[n]} \to (\mathbb{C}^2)^{(n)}$, sending I to its support $Z(I)$, is well defined and proper. It is an isomorphism precisely on the set $(\mathbb{C}^2)^{(n)}_{reg}$ corresponding to cycles

$p_1+\ldots+p_n$ consisting of n distinct points, as in this case there is only one possible scheme structure. Let $(x_1,y_1),\ldots,(x_n,y_n)$ be coordinates on $(\mathbb{C}^2)^n$. The form $\sum_k dx_k\wedge dy_k$ on $(\mathbb{C}^2)^n$ is $\mathcal{S}_n$-invariant and descends to a closed and nondegenerate form on $(\mathbb{C}^2)^{(n)}_{reg}$. A local computation shows that its pull-back by π extends to a symplectic form on $(\mathbb{C}^2)^{[n]}$. In particular π is semismall (this can also be verified directly). The subvariety $(\mathbb{C}^2)^{[n]}_0$ of subschemes supported at 0 is called the punctual Hilbert scheme of length n. Its points parametrize the n-dimensional quotient rings of $\mathbb{C}[X,Y]/(X,Y)^{n+1}$. These punctual Hilbert schemes have been studied in depth; see [103, 28], for example. They are irreducible, of dimension $n-1$, and admit a disjoint-union decomposition into affine spaces. Clearly, $(\mathbb{C}^2)^{[n]}_0\simeq(\pi^{-1}(np))_{red}$, for every $p\in\mathbb{C}^2$. Similarly, if $Z:=\sum_k n_kp_k$ with $p_i\neq p_j$ for all $i\neq j$, then $(\pi^{-1}(Z))_{red}\simeq\prod_i(\mathbb{C}^2)^{[n_i]}_0$. The construction can be globalized, in the sense that, for any nonsingular surface S, the Hilbert scheme $S^{[n]}$ is nonsingular and there is a map $\pi:S^{[n]}\to S^{(n)}$ which is semismall, and locally, in the analytic topology, isomorphic to $\pi:(\mathbb{C}^2)^{[n]}\to(\mathbb{C}^2)^{(n)}$. There also exists a version of $S^{[n]}$ for a symplectic manifold S of real dimension four, which was defined and investigated by C. Voisin in [176].

To describe the strata of the map π, we denote by $\mathfrak{P}_n$ the set of partitions of the natural number n. Let $\nu=(\nu_1,\ldots,\nu_{l(\nu)})\in\mathfrak{P}_n$, so that $\nu_1\geq\nu_2\geq\ldots\geq\nu_{l(\nu)}$ and $\sum_i\nu_i=n$. We will also write $\nu=1^{a_1}2^{a_2}\ldots n^{a_n}$, with $\sum ka_k=n$, where a_i is the number of times that the number i appears in the partition ν. Clearly $l(\nu)=\sum a_i$. We consider the following stratification of $S^{(n)}$: for $\nu\in\mathfrak{P}_n$ we set

$$S_{(\nu)}=\{\text{0-cycles}\subseteq S^{(n)}\text{ of type }\nu_1p_1+\ldots+\nu_{l(\nu)}p_{l(\nu)}\text{ with }p_i\neq p_j,\ \forall i\neq j\}.$$

Set $S_{[\nu]}=\pi^{-1}(S_{(\nu)})$ (with the reduced structure). The variety $S_{(\nu)}$ is nonsingular of dimension $2l(\nu)$. It can be shown that $\pi:S_{[\nu]}\to S_{(\nu)}$ is locally trivial with fiber isomorphic to the product $\prod_i(\mathbb{C}^2)^{\nu_i}_0$ of punctual Hilbert schemes. In particular, *the fibers of π are irreducible*; hence the local systems occurring in (4.2.7) are constant of rank one. Furthermore, the closures $\overline{S}_{(\nu)}$ and their desingularization can be explicitly determined. If ν and μ are two partitions, we say that $\mu\leq\nu$ if there exists a decomposition $I_1,\ldots,I_{l(\mu)}$ of the set $\{1,\ldots,l(\nu)\}$ such that $\mu_1=\sum_{i\in I_1}\nu_i,\ldots,\mu_{l(\mu)}=\sum_{i\in I_{l(\mu)}}\nu_i$. Then

$$\overline{S}_{(\nu)}=\coprod_{\mu\leq\nu}S_{(\mu)}.$$

This reflects just the fact that a cycle $\sum\nu_ip_i\in S_{(\nu)}$ can degenerate to a cycle in which some of the p_i's come together. If $\nu=1^{a_1}2^{a_2}\ldots n^{a_n}$, we set $S^{(\nu)}=\prod_i S^{(a_i)}$ (product of symmetric products). The variety $S^{(\nu)}$ has dimension $2l(\nu)$, and there is a natural *finite* map $\nu:S^{(\nu)}\to\overline{S}_{(\nu)}$, which is an isomorphism when restricted to $\nu^{-1}(S_{(\nu)})$. Since $S^{(\nu)}$ has only quotient singularities, it is normal, so that $\nu:S^{(\nu)}\to\overline{S}_{(\nu)}$ is the normalization map, and $IC_{\overline{S}_{(\nu)}}=\nu_*\mathbb{Q}_{S^{(\nu)}}[2l(\nu)]$.

Theorem 4.2.16. *The decomposition theorem* 4.2.7 *for $\pi:S^{[n]}\to S^{(n)}$ gives a canonical isomorphism:*

$$\pi_*\mathbb{Q}_{S^{[n]}}[2n]\ \simeq\bigoplus_{\nu\in\mathfrak{P}_n}\nu_*\mathbb{Q}_{S^{(\nu)}}[2l(\nu)]. \tag{29}$$

Taking cohomology, (29) *gives*

$$H^i(S^{[n]},\mathbb{Q}) = \bigoplus_{\nu\in\mathfrak{P}_n} H^{i+2l(\nu)-2n}(S^{(\nu)},\mathbb{Q}). \tag{30}$$

This explicit form was given by L. Göttsche and W. Soergel in [90] as an application of M. Saito's theorem [156]. Since $S^{(n)}$ is the quotient of the nonsingular variety S^n by the finite group $\mathcal{S}_n$, its *rational* cohomology $H^i(S^{(n)},\mathbb{Q})$ is just the $\mathcal{S}_n$-invariant part of $H^i(S^n,\mathbb{Q})$. In [131], MacDonald determines the dimension of such an invariant subspace. His result is more easily stated in terms of generating functions:

$$\sum \dim H^i(S^{(n)},\mathbb{Q})t^iq^n = \frac{(1+tq)^{b_1(S)}(1+t^3q)^{b_3(S)}}{(1-q)^{b_0(S)}(1-t^2q)^{b_2(S)}(1-t^4q)^{b_4(S)}}.$$

With the help of this formula and (30), we find "Göttsche's Formula" for the generating function of the Betti numbers of the Hilbert scheme:

$$\begin{aligned} &\sum_{i,n} \dim H^i(S^{[n]},\mathbb{Q})t^iq^n \\ &\quad= \prod_{m=1}^{\infty} \frac{(1+t^{2m-1}q^m)^{b_1(S)}(1+t^{2m+1}q^m)^{b_3(S)}}{(1-t^{2m-2}q^m)^{b_0(S)}(1-t^{2m}q^m)^{b_2(S)}(1-t^{2m+2}q^m)^{b_4(S)}}. \end{aligned} \tag{31}$$

Remark 4.2.17. Setting $t=-1$, we get the following simple formula for the generating function for the Euler characteristic:

$$\sum_{n=0}^{\infty} \chi(S^{[n]})q^n = \prod_{m=1}^{\infty} \frac{1}{(1-q^m)^{\chi(S)}}.$$

See [43], for a rather elementary derivation of this formula.

Göttsche's Formula appeared first in [89], following some preliminary work in the case $S=\mathbb{C}^2$ by Ellingsrud and Strømme ([71, 72]). The original proof relies on the Weil conjectures, and on a delicate counting of points over a finite field with the help of the cellular structure of the punctual Hilbert scheme following from Ellingsrud and Strømme's results.

4.3. **The functions-sheaves dictionary and geometrization.** In §4.4 and §4.5 we discuss two rather deep applications of the decomposition theorem to geometric representation theory. Even though the applications can be stated and proved within the realm of complex geometry, they have been inspired by the Grothendieck philosophy ([95]) (see also [122], §1.1) of the *dictionnaire fonctions-faisceaux* for varieties defined over finite fields. This section is devoted to a brief explanation of this philosophy. The reader who is unfamiliar with algebraic geometry over finite fields may look at §3.1.

Suppose that X_0 is a variety defined over a finite field $\mathbb{F}_q$ of cardinality q. Associated with any complex of l-adic sheaves K_0 on X_0, there is the function $t_{K_0}: X_0(\mathbb{F}_q) \to \overline{\mathbb{Q}}_l$:

$$t_{K_0}(x) = \sum_i (-1)^i \mathrm{Trace}(Fr: \mathcal{H}^i_x(K) \to \mathcal{H}^i_x(K)),$$

where Fr is the Frobenius endomorphism of $\mathcal{H}^i_x(K)$. This function is additive with respect to distinguished triangles in $D^b_c(X_0,\overline{\mathbb{Q}}_l)$, multiplicative with respect to tensor products of complexes, compatible with pull-backs, and satisfies the Grothendieck trace formula: If $f : X_0 \to Y_0$ is a proper map of $\mathbb{F}_q$-schemes and $K_0 \in D^b_c(X_0, \overline{\mathbb{Q}}_l)$, then, for $y \in Y(\mathbb{F}_q)$,

$$t_{f_* K_0}(y) = \sum_{x \in f^{-1}(y) \cap X_0(\mathbb{F}_q)} t_{K_0}(x).$$

Since l-adic sheaves on X_0 yield these trace-like functions, one may think of replacing certain classes of *functions* on X_0 with (complexes of) l-adic *sheaves* on X_0.

The philosophy of *geometrization* is rooted in the fact that quite often functions arising from representation theory or combinatorics can be interpreted as associated with sheaves (often perverse sheaves) on algebraic varieties, and theorems about such functions become consequences of theorems about the corresponding sheaves.

If the cohomology sheaves of K_0 are zero in odd degree and, for every i, the eigenvalues of Frobenius (not just their absolute values!) on $\mathcal{H}^{2i}_x(K_0)$ are equal to q^i, then the function t_{K_0} satisfies the relation $t_{K_0}(x) = \sum_i \dim \mathcal{H}^{2i}_x(K_0) q^i$.

We can modify this formula so that it makes sense for a constructible complex of sheaves K on a complex algebraic variety X. We do so by considering q to be a free variable. If the stalk cohomology vanishes in all odd degrees, then we obtain a Poincaré-like polynomial for K.

This is the case in the two examples we discuss in §4.4 and §4.5.

In §4.4 we show that the Kazhdan-Lusztig polynomials, which are associated in a purely combinatorial way (see (33)) with the Weyl group W of an algebraic group G, may be interpreted, via the functions-sheaves dictionary, as the Poincaré-like polynomials of the intersection complexes of Schubert varieties in the flag variety G/B of G. This fact allows a geometric interpretation of the Hecke algebra of W as an algebra of equivariant perverse sheaves on the flag varieties.

Similarly, in §4.5 we treat the case of a certain class of functions arising from the classical Satake isomorphism and which are associated, via the functions-sheaves dictionary, with the intersection complexes of certain subvarieties of the (infinite-dimensional) affine Grassmannian. This leads to a geometrization of the classical Satake isomorphism.

In both of these situations, the strategy towards geometrization is similar. We start with an algebra of functions on a group G with some invariance property. For instance, in the case treated in §4.4, the group is a Chevalley group and the functions are the left and right invariants with respect to a fixed Borel subgroup, and in the case treated in §4.5, they are the functions on an algebraic group over a local field which are left and right invariant with respect to the maximal compact subgroup of points over its ring of integers. This algebra has a natural basis, consisting of characteristic functions of double cosets, which correspond, via the functions-sheaves dictionary, to the constant sheaves concentrated on some subvarieties of a variety associated with G, i.e., the flag variety in the case of §4.4 and the affine Grassmannian in the case of §4.5. In each of the two situations, there is another basis which is more significant from the group-theoretic point of view, as it carries representation-theoretic information: it affords a description of representations of the Hecke algebra via the W-graph in the first case, and it describes the weight decomposition of the representations of the Langlands dual group in the second. The matrix relating the natural and the group-theoretic bases singles out a set of functions, the Kazhdan-Lusztig polynomials in §4.4 and the functions of formula (35) in §4.5.

The upshot is that in both cases it turns out that these functions are those associated, via the functions-sheaves dictionary, with the *intersection complexes* of the aforementioned subvarieties.

In §4.4 and §4.5, the main role is played by certain G-equivariant perverse sheaves and by the notion of l-adic purity. The decomposition theorem allows us to greatly simplify the arguments and to clarify the overall picture.

4.4. **Schubert varieties and Kazhdan-Lusztig polynomials.** The connection between the Kazhdan-Lusztig polynomials associated with the Weyl group of a semisimple linear algebraic group and the intersection cohomology groups of the Schubert varieties of the associated flag variety played an important role in the development of the theory of perverse sheaves. This connection was worked out by D. Kazhdan and G. Lusztig ([118], [119]), following discussions with R. MacPherson and P. Deligne.

We quickly review the basic definitions in the more general framework of Coxeter groups; see [102] for more details on this beautiful subject. Let (W, S) be a Coxeter group, that is, a group W with a set of generators S which satisfy relations $(ss')^{m(ss')} = 1$ with $m(s,s) = 1$ and $m(s,s') \geq 2$ if $s \neq s'$. Any element $w \in W$ has an expression $w = s_1 \cdots s_n$ with $s_i \in S$, and the *length* $l(w)$ of $w \in W$ is the minimal number of s_i's appearing in such an expression. For the definition of the Bruhat order, a partial order on W compatible with lengths, see [102], 5.9.

Example 4.4.1. Let $W = \mathcal{S}_{n+1}$, the symmetric group. Then the set of transpositions $s_i = (i, i+1)$ yields a set of generators $S = \{s_1, \ldots, s_n\}$.

A basic object associated with (W, S) is the Hecke algebra $\mathfrak{H}$. It is a free module over the ring $\mathbb{Z}[q^{1/2}, q^{-1/2}]$ with basis $\{T_w\}_{w \in W}$ and ring structure

$$T_w T_{w'} = T_{ww'} \text{ if } l(ww') = l(w) + l(w'),$$
$$T_s T_w = (q-1)T_w + qT_{sw} \text{ if } l(sw) < l(w).$$

As the following two examples show, Hecke algebras often arise as convolution algebras in Lie theory. Recall that, given a locally compact topological group G with Haar measure dg, the convolution product of two compactly supported measurable functions $f_1, f_2 : G \longrightarrow \mathbb{Z}$ is defined as

$$f_1 * f_2(h) = \int_G f_1(g) f_2(g^{-1}h) dg. \tag{32}$$

In the case of a finite group, the ring of $\mathbb{Z}$-valued functions with respect to the convolution product is thus canonically isomorphic to the group ring $\mathbb{Z}[G]$.

Example 4.4.2. Let G_q be a Chevalley group over the finite field with q elements $\mathbb{F}_q$, e.g., the general linear group $GL_n(\mathbb{F}_q)$, the symplectic group $Sp_{2n}(\mathbb{F}_q)$ or the orthogonal group $O_n(\mathbb{F}_q)$. Let $B_q \subseteq G_q$ be a Borel subgroup, and let W be the Weyl group. We consider functions $f : G \longrightarrow \mathbb{Z}$ which are left and right B_q-invariant, that is, such that $f(b_1 g b_2) = f(g)$ for all $b_1, b_2 \in B_q$ and $g \in G_q$. The convolution of two such functions is still left and right B_q-invariant, and the corresponding algebra $\mathbb{Z}[B_q \backslash G_q / B_q]$ is generated by the characteristic functions of the double B_q-cosets.

In [106], Iwahori proved that the Bruhat decomposition

$$G_q = \coprod_{w \in W} B_q w B_q$$

determines an algebra isomorphism between $\mathbb{Z}[B_q \backslash G_q / B_q]$, and the Hecke algebra $\mathfrak{H}$ of W, where the indeterminate q is specialized to the cardinality of the field. The survey [42] gives a useful summary of the properties of this algebra and its relevance to the representation theory of groups of Lie type.

Example 4.4.3. Let $\mathcal{K}$ be a local field and $\mathcal{O}$ be its ring of integers, and denote by π a generator of the unique maximal ideal $\mathfrak{p}$ of $\mathcal{O}$. e.g., $\mathcal{K} = \mathbb{Q}_p$, $\mathcal{O} = \mathbb{Z}_p$, and $\pi = p \in \mathbb{Z}_p$, or $\mathcal{K} = \mathbb{F}_q((T))$, the field of formal Laurent series with coefficients in a finite field, $\mathcal{O} = \mathbb{F}_q[[T]]$, and $\pi = T$. Denote by q the cardinality of the residue field $k = \mathcal{O}/\pi$. Let G be a simply connected reductive group *split* over $\mathcal{K}$; that is, G contains a maximal torus T whose set of $\mathcal{K}$-points is $T(\mathcal{K}) = (\mathcal{K}^*)^r$. Let W^{aff} be its affine Weyl group, i.e., the semidirect product of W with the co-root lattice of G; see [102], §4.2. We call the map $\pi : G(\mathcal{O}) \to G(k)$ the "reduction mod-$\mathfrak{p}$" map. Let $B' := \pi^{-1}(B)$ be the inverse image of a Borel subgroup of $G(k)$. For instance, if $G = SL_2$ with the usual choice of a positive root, and $\mathcal{K} = \mathbb{Q}_p$, then the "Iwahori subgroup" B' consists of matrices in $SL_2(\mathbb{Z}_p)$ whose entry on the upper right corner is a multiple of p. Iwahori and Matsumoto [107] proved that the algebra $\mathbb{Z}[B' \backslash G(\mathcal{K})/B']$, generated by the characteristic functions of the double B'-cosets, endowed with the convolution product, is isomorphic to the Hecke algebra for W^{aff}. As in Example 4.4.2, the double B'-cosets are parameterized, via a Bruhat-type decomposition, by W^{aff}, and the basis T_w of their characteristic functions satisfies the two defining relations of the Hecke algebra of W^{aff}. The closely related "spherical Hecke algebra" will be discussed in §4.5, in connection with the geometric Satake isomorphism.

It follows from the second defining relation of the Hecke algebra that T_s is invertible for $s \in S : T_s^{-1} = q^{-1}(T_s - (q-1)T_e)$. This implies that T_w is invertible for all w.

The algebra $\mathfrak{H}$ admits two commuting involutions ι and σ, defined by

$$\iota(q^{1/2}) = q^{-1/2},\ \iota(T_w) = T_{w^{-1}}^{-1}, \quad \text{and} \quad \sigma(q^{1/2}) = q^{-1/2},\ \sigma(T_w) = (-1/q)^{l(w)} T_w.$$

The following is proved in [118]:

Theorem 4.4.4. *There exists a unique $\mathbb{Z}[q^{1/2}, q^{-1/2}]$-basis $\{C_w\}$ of $\mathfrak{H}$ with the following properties:*

$$\iota(C_w) = C_w, \qquad C_w = (-1)^{l(w)} q^{l(w)/2} \sum_{v \leq w} (-q)^{-l(v)} P_{v,w}(q^{-1}) T_v \tag{33}$$

with $P_{v,w} \in \mathbb{Z}[q]$ of degree at most $\frac{1}{2}(l(w) - l(v) - 1)$, if $v < w$, and $P_{w,w} = 1$.

The polynomials $P_{v,w}$ are called the *Kazhdan-Lusztig polynomials of* (W, S).

Remark 4.4.5. For $s \in S$, we have that $C_s = q^{-1/2}(T_s - qT_e)$ satisfies (33); hence $P_{s,s} = P_{e,s} = 1$. A direct computation shows that if $W = \mathcal{S}_3$, then $P_{v,w} = 1$ for all v, w. In contrast, if $W = \mathcal{S}_4$, then $P_{s_1 s_3,\, s_1 s_3 s_2 s_3 s_1} = P_{s_2,\, s_2 s_1 s_3 s_2} = 1 + q$.

Let G be a semisimple linear algebraic group, T be a maximal torus, and $W = N(T)/T$ be the Weyl group. Choose a system of simple roots. Each simple root yields a reflection about the hyperplane associated with the root. The set of these reflections is known to generate W. Let B be the Borel subgroup containing T and associated with the choice of the simple roots: this means that the Lie algebra of B is spanned by the Lie algebra of T and the positive roots spaces. If $w \in W$, then we denote a representative of w in $N(T)$ by the same letter.

The flag variety $X = G/B$ parametrizes the Borel subgroups via the map $gB \to gBg^{-1}$. The B-action on X gives the "Bruhat decomposition" by B-orbits $X = \coprod_{w \in W} X_w$. The Schubert cell X_w is the B-orbit of wB. It is well known, see [18], that $X_w \simeq \mathbb{C}^{l(w)}$ and $\overline{X}_w = \coprod_{v \leq w} X_v$, where $\leq$ is the Bruhat ordering. Hence the Schubert variety $\overline{X}_w$ is endowed with a natural B-invariant cell decomposition.

Example 4.4.6. Let $G = SL_{n+1}$, B be the subgroup of upper triangular matrices, T be the subgroup of diagonal matrices. Then $W \simeq \mathcal{S}_{n+1}$, and the choice of B corresponds to $S = \{s_1, \dots s_n\}$ as in Example 4.4.1. Clearly $X_e = \overline{X}_e$ is the point B, and $\overline{X}_{w_0} = X$, if w_0 denotes the longest element of W. If $s \in S$, then $\overline{X}_s \simeq \mathbb{P}^1$. If $\{o\} \subseteq \mathbb{C}^0 \subseteq \mathbb{C}^1 \subseteq \dots \subseteq \mathbb{C}^n$ is the flag determined by the canonical basis of $\mathbb{C}^n$, then $\overline{X}_{s_i}$ parameterizes the flags $\{o\} \subseteq V_1 \subseteq \dots \subseteq V_{n-1} \subseteq \mathbb{C}^n$ such that $V_k = \mathbb{C}^k$ for all $k \neq i$. One such flag is determined by the line $V_i/V_{i-1} \subseteq V_{i+1}/V_{i-1}$. If $l(w) \geq 2$, then the Schubert variety $\overline{X}_w$ is, in general, singular. The flags $\mathbb{V} = \{o\} \subseteq V_1 \subseteq \dots \subseteq V_{n-1} \subseteq \mathbb{C}^n$ in a Schubert cell X_w can be described in terms of the dimension of the intersections $V_i \cap \mathbb{C}^j$ as follows:

$$X_w = \{\mathbb{V} : \dim V_i \cap \mathbb{C}^j = w_{ij}\}, \quad \text{where } w_{ij} = \sharp\{k \leq i \text{ such that } w(k) \leq j\}\}.$$

Since B acts transitively on any Schubert cell, it follows that $\dim \mathcal{H}^i(IC_{\overline{X}_w})_x$ depends only on the cell X_v containing the point x.

Set, for $v \leq w$, $h^i(\overline{X}_w)_v := \dim \mathcal{H}^i(IC_{\overline{X}_w})_x$ for x any point in X_v. Define, for $v \leq w$, the Poincaré polynomial $\widetilde{P}_{v,w}(q) = \sum_i h^{i-l(w)}(\overline{X}_w)_v q^{i/2}$.

We have the following remarkable and surprising fact, which yields a geometric interpretation of the Kazhdan-Lusztig polynomials in terms of dimensions of stalks of cohomology sheaves of intersection complexes of Schubert varieties.

Theorem 4.4.7 ([119])**.** *We have $P_{v,w}(q) = \widetilde{P}_{v,w}(q)$. In particular, if $i + l(w)$ is odd, then $\mathcal{H}^i(IC_{\overline{X}_w}) = 0$, and the coefficients of the Kazhdan-Lusztig polynomials $P_{v,w}(q)$ are nonnegative.*

Remark 4.4.8. Theorem 4.4.7 implies that $P_{v,w} = 1$ for all $v \leq w$ iff $IC_{\overline{X}_w} = \mathbb{Q}_{\overline{X}_w}[l(w)]$. This happens, for instance, for SL_3 (cf. Remark 4.4.5). The Schubert varieties of SL_3 are in fact smooth.

Remark 4.4.9. To our knowledge, there is no purely combinatorial proof of the nonnegativity of the coefficients of the Kazhdan-Lusztig polynomials. This fact illustrates the power of the geometric interpretation.

Remark 4.4.10. In the same paper [118] in which the polynomials $P_{v,w}$ are introduced, Kazhdan and Lusztig conjecture a formula, involving the values $P_{v,w}(1)$, for the multiplicity of a representation in the Jordan-Hölder sequences of Verma modules. The proofs of this conjecture, due independently to Beilinson-Bernstein ([10]) and Brylinski-Kashiwara ([31]), make essential use of the geometric interpretation given by Theorem 4.4.7 of the Kazhdan-Lusztig polynomials. See [167], §3, for the necessary definitions and a sketch of the proof.

Remark 4.4.11. Since $\dim X_v = l(v)$, the support conditions (12) of §2.1 for intersection cohomology imply that if $v < w$, then $\mathcal{H}^{i-l(w)}(IC_{\overline{X}_w})_v = 0$ for $i - l(w) \geq -l(v)$. It follows that the degree of $\widetilde{P}_{v,w}(q)$ is at most $\frac{1}{2}(l(w) - l(v) - 1)$, as required by the definition of the Kazhdan-Lusztig polynomials. Furthermore, as $(IC_{\overline{X}_w})_{|X_w} = \mathbb{Q}_{X_w}[l(w)]$, we have $P_{w,w} = 1$.

The original proof of Theorem 4.4.7, given in [119], is inspired to the "functions-sheaves dictionary" briefly discussed in §4.3, and does not use the decomposition theorem, but, rather, the purity of the intersection cohomology complex in the l-adic context (see §3.1) and the Lefschetz Trace Formula, [95]. As seen in Remark 4.4.11, the polynomials $\widetilde{P}_{v,w}$ satisfy the first property (33) on the degree. It thus remains to show the invariance under the involution ι. Kazhdan and Lusztig directly show that $\mathcal{H}^i(IC_{\overline{X}_w}) = 0$ if $i + l(w)$ is odd, and that the Frobenius map acts on $\mathcal{H}^{2i-l(w)}(IC_{\overline{X}_w})$ with eigenvalues equal to q^i, so that, up to a shift, $\widetilde{P}_{v,w}(q) = t_{IC_{\overline{X}_w}}(x)$, if $x \in X_v(\mathbb{F}_q)$. Once this is shown, the invariance under the involution ι turns out to be equivalent to the Poincaré duality theorem for intersection cohomology, §2.1.

For another approach, again based on the purity of the l-adic intersection cohomology complex, see [130].

An approach to Theorem 4.4.7 due to MacPherson, also gives a topological description of the Hecke algebra. It is based on exploiting the decomposition theorem for the Bott-Samelson variety (see [21, 67]), which is a G-equivariant resolution of a variety closely related to the Schubert cell $\overline{X}_w$.

Another proof, which still relies on applying the decomposition theorem to the resolutions of the Schubert varieties mentioned above, was later worked out by T. Haines, in [97]. It exploits the fact that the fibers of the resolution have a decomposition as a disjoint union of affine spaces. This latter approach works with the flag variety as well as with the (infinite-dimensional) affine flag variety.

4.5. **The Geometric Satake isomorphism.** We now discuss an analogue of the constructions described in §4.4, culminating in a geometrization of the spherical Hecke algebra and the Satake isomorphism. In this case, the Schubert subvarieties will be replaced by certain subvarieties $\overline{\mathrm{Orb}}_\lambda$ of the affine Grassmannian $\mathcal{GR}_G$.

Let us first recall, following the clear exposition [94], the basic statement of the classical Satake isomorphism ([161]).

Let $\mathcal{K}$, $\mathcal{O}$, $\mathfrak{p}$, π and q be as in §4.4.3. We let G be a reductive linear algebraic group split over $\mathcal{K}$. We denote by $G(\mathcal{K})$ the set of $\mathcal{K}$-points and by $K = G(\mathcal{O})$, the set of $\mathcal{O}$-points, a compact subgroup of $G(\mathcal{K})$. Similarly to Examples 4.4.2, 4.4.3, the *spherical* Hecke algebra $\mathcal{H}(G(\mathcal{K}), G(\mathcal{O}))$ is defined to be the set of $K-K$-invariant locally constant $\mathbb{Z}$-valued functions on $G(\mathcal{K})$ endowed with the convolution product (32), where the Haar measure is normalized so that the volume of K is 1.

The group $X_\bullet(T) := \mathrm{Hom}(\mathcal{K}^*, T(\mathcal{K}))$ of co-characters of T is free abelian and carries a natural action of the Weyl group W. The choice of a set of positive roots singles out a system of positive co-roots in $X_\bullet(T)$ as well as the positive chamber

$$X_\bullet(T)^+ = \{\lambda \in X_\bullet(T)\, s.t.\, \lambda(\alpha) \geq 0 \, if \, \alpha > 0\},$$

which is a fundamental domain for the action of W. Given $\lambda, \mu \in X_\bullet(T)$, we say that $\lambda \geq \mu$ if $\lambda - \mu$ is a sum of positive co-roots.

Every $\lambda \in X_\bullet(T)$ defines an element $\lambda(\pi) \in K$, and one has the following Cartan-type decomposition:

$$G = \coprod_{\lambda \in X_\bullet(T)^+} K\lambda(\pi)K.$$

The characteristic functions C_λ of the double cosets $K\lambda(\pi)K$, for $\lambda \in X_\bullet(T)^+$, give a $\mathbb{Z}$-basis of $\mathcal{H}(G,K)$. The spherical Hecke algebra is commutative.

Remark 4.5.1. For the torus T, we have $\mathcal{H}(T(\mathcal{K}), T(\mathcal{O})) \simeq \mathbb{Z}[X_\bullet(T)]$.

Example 4.5.2. Let $G = GL_n$. With the usual choice of positive roots, an element $\lambda \in X_\bullet(T)^+$ is of the form $\mathrm{diag}(t^{a_1}, \ldots t^{a_n})$, with $a_1 \geq a_2 \geq \ldots \geq a_n$. The above decomposition boils down to the fact that a matrix can be reduced to diagonal form by multiplying it on the left and on the right by elementary matrices.

The *Langlands dual* ${}^LG$ of G is the reductive group whose root datum is the co-root datum of G and whose co-root datum is the root datum of G. For a very nice description of these notions, see [166]. The representation ring $R({}^LG)$ of ${}^LG$ is isomorphic to $\mathbb{Z}[X_\bullet(T)]^W$.

Theorem 4.5.3 (The classical Satake isomorphism)**.** *There is an isomorphism of algebras:*

$$\mathcal{S} : \mathcal{H}(G(\mathcal{K}), G(\mathcal{O})) \otimes \mathbb{Z}[q^{1/2}, q^{-1/2}] \xrightarrow{\simeq} R({}^LG) \otimes \mathbb{Z}[q^{1/2}, q^{-1/2}]. \tag{34}$$

Remark 4.5.4. The $\mathbb{Z}$-module $R({}^LG)$ has a basis $[V_\lambda]$ parameterized by $\lambda \in X_\bullet(T)^+$, where V_λ is the irreducible representation with highest weight λ. It may be tempting to think that the inverse $\mathcal{S}^{-1}$ sends $[V_\lambda]$ to the characteristic function C_λ of the double coset $K\lambda(\pi)K$. However, this does not work: there exist integers $d_\lambda(\mu)$, defined for $\mu \in X_\bullet(T)^+$, with $\mu < \lambda$ such that the more complicated formula

$$\mathcal{S}^{-1}([V_\lambda]) = q^{-\rho(\lambda)}(C_\lambda + \sum_{\substack{\mu \in X_\bullet(T)^+ \\ \mu < \lambda}} d_\lambda(\mu) C_\mu), \tag{35}$$

where $\rho = (1/2)\sum_{\alpha > 0} \alpha$, holds instead.

The Satake isomorphism is remarkable in the sense that it relates G and ${}^LG$. A priori, it is unclear that the two should be related at all, beyond the defining exchanging property. The isomorphism gives, in principle, a recipe to construct the (representation ring of the) Langlands dual ${}^LG$ of G from the datum of the ring of functions on the double coset space $K\backslash G/K$.

A striking application of the theory of perverse sheaves is the "geometrization" of this isomorphism. The whole subject was started by the important work of Lusztig [127, 126]. In this work, it is shown that the Kazhdan-Lusztig polynomials associated with a group closely related to W^{aff} are the Poincaré polynomials of the intersection cohomology sheaves of the singular varieties $\overline{\mathrm{Orb}}_\lambda$, for $\lambda \in X_\bullet(T)$, inside the affine Grassmannian $\mathcal{GR}_G$ defined below, and they coincide with the weight multiplicities $d_\lambda(\mu)$ of the representation V_λ appearing in formula (35). As a consequence, he showed that, if we set $IH^*(\overline{\mathrm{Orb}}_\lambda) = \bigoplus_l IH^l(\overline{\mathrm{Orb}}_\lambda)$, then we have $\dim IH^*(\overline{\mathrm{Orb}}_\lambda) = \dim V_\lambda$ and that the tensor product operation $V_\lambda \otimes V_\nu$ correspond to a "convolution" operation $IC_{\overline{\mathrm{Orb}}_\lambda} \star IC_{\overline{\mathrm{Orb}}_\nu}$.

The geometric significance of Lusztig's result was clarified by the work of Ginzburg [83] and Mirković-Vilonen [145].

We quickly review the geometry involved, according to the paper [145]. We work over the field of complex numbers. The analogue of the coset space $G(\mathcal{K})/G(\mathcal{O}))$ of §4.4 is the affine Grassmannian, which we now introduce; see [12] for a thorough treatment. Let G be a reductive algebraic group over $\mathbb{C}$, let $\mathbb{C}[[t]]$ be the ring of formal power series and $\mathbb{C}((t))$ its fraction field of Laurent series. The quotient $\mathcal{GR}_G = G(\mathbb{C}((t)))/G(\mathbb{C}[[t]])$ is called the *affine Grassmannian*: it is an ind-variety, i.e., a countable increasing union of projective varieties.

Remark 4.5.5. Let $G = SL_n(\mathbb{C})$. The points of $\mathcal{GR}_{SL_n(\mathbb{C})}$ parameterize special lattices in the $\mathbb{C}((t))$-vector space $V = \mathbb{C}((t))^n$. A special lattice is a $\mathbb{C}[[t]]$-module $M \subseteq V$ such that $t^N\mathbb{C}[[t]]^n \subseteq M \subseteq t^{-N}\mathbb{C}[[t]]^n$ for some N, and $\bigwedge^n M = \mathbb{C}[[t]]$. The action of $SL_n(\mathbb{C}((t)))$ on the set of special lattices is transitive, and $SL_n(\mathbb{C}[[t]])$ is the stabilizer of the lattice $M = \mathbb{C}[[t]]^n$.

Remark 4.5.6. The set of points of the affine Grassmannian $\mathcal{GR}_T$ of a torus T is easily seen to be $X_\bullet(T)$ (see Remark 4.5.1). The scheme structure is somewhat subtler, as it turns out to be nonreduced.

Set $\mathcal{K} = \mathbb{C}((t))$ and $\mathcal{O} = \mathbb{C}[[t]]$. The imbedding $T \subseteq G$ of the maximal torus gives a map $\mathcal{GR}_T \to \mathcal{GR}_G$; thus, by Remark 4.5.6, we can identify $X_\bullet(T)$ with a subset of $\mathcal{GR}_G$. It turns out that the group $G(\mathcal{O})$ acts on $\mathcal{GR}_G$ with finite-dimensional orbits. We still denote by λ the point of the affine Grassmannian corresponding to $\lambda \in X_\bullet(T)$, and denote its $G(\mathcal{O})$-orbit by $\mathrm{Orb}_\lambda \subseteq \mathcal{GR}_G$ (cf. [12]).

Proposition 4.5.7 ([12], 5.3)**.** *There is a decomposition* $\mathcal{GR}_G = \coprod_{\lambda\in X_\bullet(T)^+} \mathrm{Orb}_\lambda$. *Furthermore, every orbit* Orb_λ *has the structure of a vector bundle over a rational homogeneous variety, it is connected and simply connected,*

$$\dim \mathrm{Orb}_\lambda = 2\rho(\lambda) \qquad \textit{and} \qquad \overline{\mathrm{Orb}}_\lambda = \coprod_{\mu\leq\lambda} \mathrm{Orb}_\mu.$$

Proposition 4.5.7 implies that the category $\mathcal{P}_{G(\mathcal{O})}$ of perverse sheaves which are constructible with respect to the decomposition in $G(\mathcal{O})$-orbits is generated by the intersection cohomology complexes $IC_{\overline{\mathrm{Orb}}_\lambda}$. Lusztig has proved in [127] that the cohomology sheaves $\mathcal{H}^i(IC_{\overline{\mathrm{Orb}}_\lambda})$ are different from zero only in one parity. Together with the fact that the dimensions of all $G(\mathcal{O})$-orbits in the same connected component of $\mathcal{GR}_G$ have the same parity, this implies that $\mathcal{P}_{G(\mathcal{O})}$ is a semisimple category. Its objects are automatically $G(\mathcal{O})$-equivariant perverse sheaves.

The Tannakian formalism, see [65], singles out the categories which are equivalent to categories of representations of affine group schemes, and it gives a precise prescription for reconstructing the group scheme from its category of representations. The geometrization of the Satake isomorphism essentially states that the category $\mathcal{P}_{G(\mathcal{O})}$ is equivalent to the category of representations $\mathrm{Repr}({}^LG)$ of the Langlands dual group ${}^LG$, so it yields a recipe to reconstruct this dual group. More precisely, it is necessary to endow $\mathcal{P}_{G(\mathcal{O})}$ with the structure of a rigid tensor category with a "fiber functor." Essentially, this means that there must be 1) a bilinear functor $\star : \mathcal{P}_{G(\mathcal{O})} \times \mathcal{P}_{G(\mathcal{O})} \to \mathcal{P}_{G(\mathcal{O})}$ with compatible associativity and commutativity constraints, i.e., functorial isomorphisms $A_1 \star (A_2 \star A_3) \overset{\simeq}{\to} (A_1 \star A_2) \star A_3$ and $A_1 \star A_2 \overset{\simeq}{\to} A_2 \star A_1$, and 2) an exact functor, called the fiber functor, $F : \mathcal{P}_{G(\mathcal{O})} \to \mathrm{Vect}_{\mathbb{Q}}$ which is a tensor functor; i.e., there is a functorial isomorphism $F(A_1 \star A_2) \overset{\simeq}{\to} F(A_1) \otimes F(A_2)$.

Remark 4.5.8. For the category of representations of a group, the product is given by the tensor product of representations, while the fiber functor ω associates with a representation its underlying vector space.

In fact, there exists a geometrically defined "convolution product"

$$\star \;:\; \mathcal{P}_{G(\mathcal{O})} \times \mathcal{P}_{G(\mathcal{O})} \longrightarrow \mathcal{P}_{G(\mathcal{O})}$$

with "associativity and commutativity constraints," such that the cohomology functor $H : \mathcal{P}_{G(\mathcal{O})} \to \mathrm{Vect}_{\mathbb{Q}}$ is a tensor functor. The construction of this geometric convolution product is reviewed below; see Theorem 4.5.11.

We state the geometric Satake isomorphism as follows:

Theorem 4.5.9 (**Geometric Satake isomorphism**). *There is an equivalence of tensor categories*

$$\mathcal{S}_{geom} : (\mathcal{P}_{G(\mathcal{O})}, \star, H) \xrightarrow{\simeq} (\mathrm{Rep}({}^{L}G), \otimes, \omega). \tag{36}$$

Remark 4.5.10. Nadler investigated ([148]) a subcategory of perverse sheaves on the affine Grassmannian of a real form $G_{\mathbb{R}}$ of G and proved that it is equivalent to the category of representations of a reductive subgroup ${}^{L}H$ of ${}^{L}G$. This establishes a real version of the Geometric Satake isomorphism and, as a corollary, the decomposition theorem is shown to hold for several *real* algebraic maps arising in Lie theory.

We discuss only two main points of the construction of [145], the definition of the convolution product and the use of the " semi-infinite" orbits to construct the weight functors. We omit all technical details and refer the reader to [145].

The convolution product. In the following description of the convolution product we treat the spaces involved as if they were honest varieties. See [80] for a detailed account. Let us consider the diagram:

$$\begin{tikzcd}
\mathcal{GR}_G & G(\mathcal{K}) \arrow[rr, "\pi"] & & \mathcal{GR}_G & \\
G(\mathcal{K}) \times_{G(\mathcal{O})} \mathcal{GR}_G \arrow[u, "p"] & G(\mathcal{K}) \times \mathcal{GR}_G \arrow[l, "q"'] \arrow[u] \arrow[rr, "\pi\times\mathrm{Id}"] & & \mathcal{GR}_G \times \mathcal{GR}_G \arrow[u, "p_1"'] \arrow[r, "p_2"] & \mathcal{GR}_G.
\end{tikzcd}$$

The map $q : G(\mathcal{K}) \times \mathcal{GR}_G \to G(\mathcal{K}) \times_{G(\mathcal{O})} \mathcal{GR}_G$ is the quotient map by the action of $G(\mathcal{O})$, the map $p : G(\mathcal{K}) \times_{G(\mathcal{O})} \mathcal{GR}_G \to \mathcal{GR}_G$ is the "action" map, $p(g, hG(\mathcal{O})) = ghG(\mathcal{O})$. If $A_1, A_2 \in \mathcal{P}_{G(\mathcal{O})}$, then $(\pi \times \mathrm{Id})^*(p_1^*(A_1) \otimes p_2^*(A_2))$ on $G(\mathcal{K}) \times \mathcal{GR}_G$ descends to $G(\mathcal{K}) \times_{G(\mathcal{O})} \mathcal{GR}_G$; that is, there exists a unique complex of sheaves $A_1 \widetilde{\otimes} A_2$ on $G(\mathcal{K}) \times_{G(\mathcal{O})} \mathcal{GR}_G$ with the property that $(\pi \times \mathrm{Id})^*(p_1^*(A_1) \otimes p_2^*(A_2)) = q^*(A_1 \widetilde{\otimes} A_2)$, and we set $A_1 \star A_2 := p_*(A_1 \widetilde{\otimes} A_2)$.

The following fact is referred to as "Miraculous theorem" in [12]:

Theorem 4.5.11. *If $A_1, A_2 \in \mathcal{P}_{G(\mathcal{O})}$, then $A_1 \star A_2 \in \mathcal{P}_{G(\mathcal{O})}$.*

The key reason why this theorem holds is that the map p enjoys a strong form of semismallness.

First of all the complex $A_1 \widetilde{\otimes} A_2$ is constructible with respect to the decomposition

$$G(\mathcal{K}) \times_{G(\mathcal{O})} \mathcal{GR}_G = \coprod \mathcal{S}_{\lambda,\mu} \qquad \text{with} \qquad \mathcal{S}_{\lambda,\mu} = \pi^{-1}(\mathrm{Orb}_\lambda) \times_{G(\mathcal{O})} \mathrm{Orb}_\mu.$$

Proposition 4.5.12. *The map $p : G(\mathcal{K}) \times_{G(\mathcal{O})} \mathcal{GR}_G \to \mathcal{GR}_G$ is stratified semi-small, in the sense, that for any $\mathcal{S}_{\lambda,\mu}$, the map $p_{|\overline{\mathcal{S}}_{\lambda,\mu}} : \overline{\mathcal{S}}_{\lambda,\mu} \to p(\overline{\mathcal{S}}_{\lambda,\mu})$ is semismall. As a consequence, p_* sends perverse sheaves, constructible with respect to the decomposition $\{\mathcal{S}_{\lambda,\mu}\}$, to perverse sheaves on $\mathcal{GR}_G$ constructible with respect to the decomposition $\{\mathrm{Orb}_\lambda\}$.*

Remark 4.5.13. While the "associativity constraints" of the convolution product are almost immediate from its definition, the commutativity constraints are far subtler (see [145] and also [80]).

The weight functor. The cohomology functor $H(-) := \bigoplus_l H^l(-)$ is a fiber functor for the category $\mathcal{P}_{G(\mathcal{O})}$. In particular, it is a tensor functor: $H(A_1 \star A_2) \simeq H(A_1) \otimes H(A_2)$. In order to verify this, Mirković and Vilonen decompose this functor as a direct sum of functors H_μ parameterized by $\mu \in X_\bullet(T)$. This decomposition is meant to mirror the weight decomposition of a representation of ${}^LG$. It is realized by introducing certain ind-subvarieties N_μ which have a "cellular" property with respect to any $A \in \mathcal{P}_{G(\mathcal{O})}$, in the sense that at most one compactly supported cohomology group does not vanish. Let U be the unipotent radical of the Borel group B, and let $U(\mathcal{K})$ be the corresponding subgroup of $G(\mathcal{K})$. The $U(\mathcal{K})$-orbits in the affine Grassmannian are neither of finite dimension nor of finite codimension. It can be shown that they are parameterized by $X_\bullet(T)$. If, as before, we still denote by ν the point of the affine Grassmannian corresponding to a co-character $\nu \in X_\bullet(T)$, and set $S_\nu := U(\mathcal{K})\nu$, then we have $\mathcal{GR}_G = \coprod_{\nu \in X_\bullet(T)} S_\nu$.

Proposition 4.5.14. *For any $A \in \mathcal{P}_{G(\mathcal{O})}$, we have $H^l_c(S_\nu, A) = 0$ for $l \neq 2\rho(\nu)$ and $H^{2\rho(\nu)}_c(S_\nu, IC_{\overline{\mathrm{Orb}}_\lambda})$ is canonically isomorphic to the vector space generated by the irreducible components of $\overline{\mathrm{Orb}}_\lambda \cap S_\nu$. In particular, the functor $H^{2\rho(\nu)}_c(S_\nu, -) : \mathcal{P}_{G(\mathcal{O})} \to \mathrm{Vect}_{\mathbb{Q}}$ sending $A \in \mathcal{P}_{G(\mathcal{O})}$ to $H^{2\rho(\nu)}_c(S_\nu, A)$ is exact, and*

$$H(A) \;:= \bigoplus_{l\in\mathbb{Z}} H(\mathcal{GR}_G, A) = \bigoplus_{\nu \in X_\bullet(T)} H^{2\rho(\nu)}_c(S_\nu, A).$$

Remark 4.5.15. Let $A \in \mathcal{P}_{G(\mathcal{O})}$. Since in the equivalence of categories of Theorem 4.5.9, the fiber functors H correspond to ω, the decomposition

$$H(A) \;= \bigoplus_{\nu \in X_\bullet(T)} H^{2\rho(\nu)}_c(S_\nu, A)$$

of Proposition 4.5.14 of the cohomology of A must reflect a decomposition of the underlying vector space of the representation $\mathcal{S}_{geom}(A)$. In fact, this is the weight decomposition of the corresponding representation of ${}^LG$.

An aspect of the Geometric Satake correspondence which we find particularly beautiful is that, up to a renormalization, the intersection cohomology complex $IC_{\overline{\mathrm{Orb}}_\lambda}$ corresponds, via the Geometric Satake isomorphism, to the irreducible representation $V(\lambda)$ of ${}^LG$ with highest weight λ. This explains (see Remark 4.5.4) why the class of $V(\lambda)$ is not easily expressed in terms of the characteristic function C_λ of the double coset $K\lambda(\pi)K$, which corresponds, in the function-sheaves dictionary of 4.3, to the constant sheaf on $\overline{\mathrm{Orb}}_\lambda$, and once again emphasizes the fundamental nature of intersection cohomology. Furthermore, in view of Proposition 4.5.14, the irreducible components of $\overline{\mathrm{Orb}}_\lambda \cap S_\nu$ as ν varies in $X_\bullet(T)$, give a canonical basis for $V(\lambda)$. These components are now called *Mirković-Vilonen cycles.*

The classical Satake isomorphism 4.5.3 for $\mathcal{K} = \mathbb{F}_q((T))$ may be recovered from the geometric Satake isomorphism 4.5.9 by considering the Grothendieck group of the two tensor categories. In fact, the Grothendieck ring of the category $\mathrm{Repr}(^L G)$ is the representation ring $R(^L G)$, while the functions-sheaves dictionary identifies the Grothendieck ring of $\mathcal{P}_{G(\mathcal{O})}$ with the spherical Hecke algebra $\mathcal{H}(G(\mathcal{K}), G(\mathcal{O}))$.

4.6. **Ngô's support theorem.** We thank G. Laumon and B.C. Ngô for very useful conversations. The paper [152] is devoted to the proof of the fundamental lemma in the Langland's program, a long-standing and deep conjecture concerning Lie groups. For its complexity, depth and wealth of applications to representation theory, this paper deserves a separate treatment, which we do not provide here. In this section, instead, we give a brief and rough discussion of B.C. Ngô's *support theorem* ([152], Theorem 7.1.13). This result, which we state in a slightly weaker form in Theorem 4.6.2, can be stated and proved without any reference to the context of the Langlands program, and it is of great independent geometrical interest. Under the favourable assumptions which are explained in the sections that follow, it gives a precise characterization of the supports of the perverse sheaves which enter the decomposition theorem for a map $f : M \to S$ acted upon in a fiber-preserving manner by a family of commutative algebraic groups $g : P \to S$. This seems to be one of the first cases in which the decomposition theorem is studied in depth in the context of a nongenerically finite map, i.e. of a map with large fibers. For expository reasons, we state these results over the complex numbers, even though the main use in [152] is in the l-adic context over a finite field.

The determination of the simple summands $IC_{\overline{Y_a}}(L_a)$ appearing in the decomposition theorem (9) is a difficult problem. The determination of the supports $\overline{Y_a}$ does not seem to be easier. In fact, consider Examples 1.8.2, 1.8.3: the vertex of the cone is certainly a special locus in both cases; however, it appears as a support of a summand in the decomposition theorem only in Example 1.8.2.

One important ingredient of Ngo's proof of the support theorem is the following result of Goresky and MacPherson, which, in the case of equidimensional maps, yields an a priori constraint on the codimension of subvarieties supporting simple summands in the decomposition theorem. The proof is a simple and elegant application of the symmetry (10) arising from Poincaré-Verdier duality, and can be found in [152], Appendice A, Théorèmes 2 and 3.

Theorem 4.6.1. *Let $f : X \to Y$ be a proper map of algebraic varieties, with X nonsingular. Assume that all the fibers of f have the same dimension d. Assume that $Z \subseteq Y$ is an irreducible subvariety which is the support of a nonzero summand appearing in the decompositon theorem* (9)*. Then*

$$\mathrm{codim}\,(Z) \leq d.$$

If, in addition, the fibers are irreducible, then one has strict inequality in the above.

The basic idea in the proof is that a larger codimension, coupled with duality, would force the corresponding summand to contribute a nontrivial summand to the direct image sheaf $R^j f_* \mathbb{Q}_X$ for $j > 2d$, contradicting the fact that the fibers have dimension d.

In order to state Ngô's support theorem, let us fix some notation.

Let $f : M \to S$ be a proper and flat map of relative dimension d with reduced fibers and where M and S are smooth irreducible varieties. The map f is assumed to be endowed with an action of a commutative group scheme $g : P \to S$ of relative dimension d. A group scheme is a map $g : P \to S$ together with S-maps $e : S \to P$, $m : P \times_S P \to P$ and $\iota : P \to P$ that satisfy the usual axioms of a group. Each fiber $g^{-1}(s)$ is an algebraic group, and a group scheme can be seen as a family of groups. In this context, an action is an S-map $a : P \times_S M \to M$ commuting with the projections to S that satisfy the usual requirements of an action, suitably modified to the "relative to S" situation.

Let $g : P \to S$ be as above and *with connected fibres*, and let $s \in S$. By a classical result of Chevalley, there is a canonical exact sequence $1 \to R_s \to P_s \to A_s \to 1$ of algebraic groups, where R_s is affine (thus a product $\mathbb{C}^{\alpha_s} \times \mathbb{C}^{*\mu_s}$ of additive and multiplicative groups), and A_s is an abelian variety. The function $\delta : S \to \mathbb{N}$, $s \mapsto \delta_s := \dim R_s = \alpha_s + \mu_s$ is well defined. This function is upper-semicontinuous; i.e., it jumps up on Zariski-closed subsets. In particular, there is a partition of $S = \coprod_{\delta \geq 0} S_\delta$ into locally closed subvarieties where the invariant δ is constant. We assume furthermore that P *acts with affine stabilizers*: for any $m \in M$, the isotropy subgroup of m is an affine subgroup of $P_{f(m)}$.

We need the two notions of δ-regularity and of polarizability. An S-group scheme $g : P \to S$ as above is *δ-regular* if

$$\operatorname{codim}_S(S_\delta) \geq \delta.$$

The Tate sheaf is the sheaf

$$T(P) := R^{2d-1} g_! \mathbb{Q}$$

whose stalk at $s \in S$ is, by base change, the homology group $H_1(P_s)$. We say that $T_{\mathbb{Q}}(P)$ is *polarizable* if there is an alternating bilinear pairing

$$T(P) \otimes T(P) \longrightarrow \mathbb{Q}_S$$

that factors through $H_1(A_s)$ at every point and induces on it a perfect pairing.

We can now state the following

Theorem 4.6.2 (**Ngô support theorem**). *Let $f : M \to S$, $g : P \to S$ be as above. Assume that $P \to S$ is δ-regular and that $T(P)$ is polarizable. A closed irreducible subvariety $Z \subseteq S$ is the support of a nontrivial simple summand appearing in the decomposition theorem for $f : M \to S$ if and only if there is a Zariski dense open subvariety $Z^0 \subseteq Z$ such that the sheaf $R^{2d} f_* \mathbb{Q}$ is locally constant on Z^0 and Z is maximal with respect to this property.*

Remark 4.6.3. Theorem 4.6.2 is applied to the case when $f : M \to S$ is a suitable open subset of the Hitchin fibration associated with a Lie group. The hypothesis of δ-regularity is verified with the aid of Riemann-Roch and of the deformation theory of Higgs bundles. Over the complex numbers an infinitesimal argument shows that a group scheme (variety) associated with a completely integrable algebraic system is always δ-regular. The hypothesis of polarizability is verified using the classical Weil pairing. See [152].

The statement of the support theorem, is remarkable because it tells us where to look for the supports of the summands of the decomposition theorem: they are those varieties' closures of (maximal) parts of S over which the single sheaf $R^{2d} f_* \mathbb{Q}$ is locally constant. On the other hand, since the fibers are assumed to be reduced, the sheaf $R^{2d} f_* \mathbb{Q}$ is the linearization of the sheaf of finite sets given by the irreducible components of the fibers of f. This fact makes the determination of these supports an approachable problem. For example, suppose that the map f has irreducible fibres; denote by $j : S_{reg} \to S$ the imbedding of the open set of regular values of f and by R^i the local systems $R^i f_* \mathbb{Q}$ on S_{reg}. Then $f_*\mathbb{Q}[\dim M] = \bigoplus_i IC(R^i)[d-i]$; that is, there are no summands beyond those which are determined by the "fibration part".

4.7. **Decomposition up to homological cobordism and signature.** We want to mention, without any detail, a purely topological counterpart of the decomposition theorem. Recall that this result holds only in the algebraic context; e.g., it fails for proper holomorphic maps of complex manifolds.

In the topological context, Cappell and Shaneson [33] introduce a notion of cobordism for complexes of sheaves and prove a general topological result for maps between Whitney stratified space with only even codimension strata that in the case of a proper algebraic map $f : X \to Y$ identifies, up to cobordism, $f_* IC_X$ with ${}^{\mathfrak{p}}\mathcal{H}^0(f_* IC_X)$ and its splitting as in the decomposition theorem. For a related question, see [88], D., Problem 6.

The decomposition up to cobordism is sufficient to provide exact formulae for many topological invariants, such as Goresky-MacPherson L-classes and signature, thus generalizing the classical Chern-Hirzebruch-Serre multiplicativity property of the signature for smooth fiber bundles with no monodromy to the case of stratified maps (see [34, 35, 163]).

In the case of complex algebraic varieties, one may also look at the MacPherson Chern classes [132], the Baum-Fulton-MacPherson Todd classes [6], the homology Hirzebruch classes [25, 37] and their associated Hodge-genera defined in terms of the mixed Hodge structures on the (intersection) cohomology groups. The papers [35, 36, 37] provide Hodge-theoretic applications of the above topological stratified multiplicative formulæ. For a survey, see [137].

These results yield topological and analytic constraints on the singularities of complex algebraic maps. In the case of maps of projective varieties, these Hodge-theoretic formulæ are proved using the decomposition theorem, especially the identification in [51] of the local systems appearing in the decomposition combined with the Hodge-theoretic aspects of the decomposition theorem in [54]. For noncompact varieties, the authors use the functorial calculus on the Grothendieck groups of Saito's algebraic mixed Hodge modules.

4.8. **Further developments and applications. Toric varieties and polytopes.** There exist polytopes that are not combinatorially equivalent to any rational polytope, and the formula for the generalized h-polynomial makes sense also in this case, even though there is no toric variety associated with it. It is thus natural to ask whether the properties of the h-polynomial reflecting the Poincaré duality and the hard Lefschetz theorem hold more generally for any polytope.

In order to study this sort of questions, P. Bressler and V. Lunts have developed a theory of sheaves on the poset associated with the polytope P, or more generally to a fan; see [26]. Passing to the corresponding derived category, they define an intersection cohomology complex and prove the analogue of the decomposition theorem for it, as well as the equivariant version.

By building on their foundational work, K. Karu proved in [112] that the hard Lefschetz property and the Hodge-Riemann relations hold for every, i.e., not necessarily rational, polytope. Different proofs, each one shedding new light on interesting combinatorial phenomena, have then been given by Bressler-Lunts in [27] and by Barthel-Brasselet-Fieseler-Kaup in [4]. Another example of an application of the methods of intersection cohomology to the combinatorics of polytopes is the solution, due to T. Braden and R. MacPherson of a conjecture of G. Kalai concerning the behavior of the g-polynomial of a face with respect to the g-polynomial of the whole polytope. See [24] and the survey [23].

The Hilbert scheme of points on a surface. Vafa and Witten noticed in [170] that Göttsche's Formula (31) suggests a representation-theoretic structure underlying the direct sum $\bigoplus_{i,n} H^i(S^{[n]})$. Namely, this space should be an irreducible highest weight module over the infinite-dimensional Heisenberg-Clifford super Lie algebra, with highest weight vector the generator of $H^0(S^{[0]})$. H. Nakajima and, independently, I. Grojnowski took up the suggestion in [151, 93] (see also the lecture notes [149]) and realized this structure by a set of correspondences relating Hilbert schemes of different lengths.

An elementary proof of Göttsche's formula stemming from this circle of ideas was given in [47]. The papers [49, 50] prove, in two different ways, a motivic version of the decomposition theorem (29) for the map $\pi : S^{[n]} \to S^{(n)}$ exhibiting an equality

$$(S^{[n]}, \Delta, 2n) = \sum_{\nu \in \mathfrak{P}_n} (S^{l(\nu)}, P_\nu, 2l(\nu))$$

of Chow motives with rational coefficients. In this formula, P_ν denotes the projector associated with the action of the group $\prod \mathcal{S}_{a_i}$ on $S^{l(\nu)}$. Two related examples, still admitting a semismall contraction, are the nested Hilbert scheme $S^{[n,n+1]}$, whose points are couples $(Z, Z') \in S^{[n]} \times S^{[n+1]}$ such that $Z \subseteq Z'$, and the parabolic Hilbert scheme; see [50] and its Appendix for details.

The Geometric Satake isomorphism. The decomposition theorem, applied to the stratified semismall map p used to define the convolution, gives a decomposition

$$IC_{\overline{\mathrm{Orb}_\lambda}} \star IC_{\overline{\mathrm{Orb}_\mu}} = \bigoplus_{\nu} IC_{\overline{\mathrm{Orb}_\nu}} \otimes F_\nu,$$

where F_ν is the vector space generated by the relevant irreducible components of the fibres of p (see (24) and note that the strata are simply connected). This decomposition mirrors, on the geometric side, the Clebsch-Gordan decomposition $V(\lambda) \otimes V(\mu) = \bigoplus_\nu V(\nu) \otimes F_\nu$. The irreducible components of the fibres were shown to be Mirković-Vilonen cycles in [1]. A combinatorial study of Mirković-Vilonen cycles is made possible by letting the maximal torus T act on them. The action is Hamiltonian and its image by the moment map is a polytope. The so-obtained *Mirković-Vilonen polytopes* are investigated in, for instance, [2, 111].

Other applications. The examples discussed in this section are far from exhausting the range of applications of the theory of perverse sheaves. We suggest G. Lusztig's [128], T.A. Springer's [167], and N. Chriss and V. Ginzburg's [39] for further applications and for more details, including motivation and references, about some of the examples discussed here in connection with representation theory.

For lack of space and competence, we have not discussed many important examples, such as the proof of the Kazhdan-Lusztig conjectures and the applications of the geometric Fourier transform.

The most dramatic occurrence of the functions-sheaves dictionary, and one of the reasons for the importance of perverse sheaves in representation theory, is the geometrization of the notion of *automorphic form* in the geometric Langlands program; for details, see for instance [76], §3.3, or [79]. Coarsely speaking, an (unramified) automorphic form is a function on the "adèlic quotient" $GL_n(F)\backslash GL_n(\mathbb{A}_F)/GL_n(\mathcal{O})$, where F is the field of rational functions of an algebraic curve X defined over a finite field $\mathbb{F}_q$, $\mathbb{A}_F$ is the ring of adèles of F, and $\mathcal{O} = \prod_{x \in X} \mathcal{O}_x$. The function must also satisfy some other property, such as that of being an eigenvector for the unramified Hecke algebra. A theorem of A. Weil gives an interpretation of the adèlic quotient as the set of points of the moduli stack of vector bundles on X. Hence, by the function-sheaves dictionary, an automorphic form should correspond to a perverse sheaf on this moduli stack, and the important condition that the automorphic form be a Hecke eigenvector can also be interpreted geometrically by introducing the notion of a Hecke eigensheaf.

5. Appendices

5.1. **Algebraic varieties.** The precise definitions of varieties and maps in algebraic geometry are quite lengthy. Luckily, in order to understand the statement of the decomposition theorem, as well as some of its applications, it is often sufficient to deal with quasi-projective varieties and the maps between them. Let us explain a little bit this terminology, without being too formal. A *projective* variety is an algebraic variety that admits an embedding in some projective space $\mathbb{P}^N$ as the zero set of finitely many homogeneous equations in $N+1$ variables. A *quasi-projective* variety is an algebraic variety that admits an embedding in some projective space as the difference set of two projective varieties. There are algebraic varieties that are not quasi-projective. An *affine* variety is a variety that can be viewed as the zero set in some affine space $\mathbb{A}^N$ of finitely many polynomial in N variables. An affine variety is clearly quasi-projective; the converse does not hold, e.g., $\mathbb{A}^2 \setminus \{(0,0)\}$. We also have the notions of *subvariety* and *product varieties.* A map of algebraic varieties $f : X \to Y$, or simply a *map*, is a map of the underlying sets whose graph is an algebraic subvariety of the product variety $X \times Y$. A complex algebraic variety carries two interesting topologies: the *Euclidean* (or classical) topology and the coarser *Zariski* topology. Let us discuss these two topologies in the case of a quasi-projective variety embedded in a projective space, $X \subseteq \mathbb{P}^N$: the Euclidean topology is the topology induced on X by the complex manifold topology on $\mathbb{P}^N$; the Zariski topology is the topology with closed sets given by zero sets on X of finitely many homogeneous polynomials in $N+1$ variables. A *closed* (*open*, resp.) subvariety is a closed (open, resp.) subset for the Zariski topology. A map of algebraic varieties is *proper* (in the sense of algebraic geometry) if it is separated and universally closed for the Zariski topology; luckily, this happens if and only if the map is proper for the Euclidean topology. In particular, a map of projective varieties is always proper. An algebraic variety X is *reducible* if it is the union $X = X' \cup X''$ of two closed algebraic subvarieties with $X', X'' \neq X$, and it is *irreducible* otherwise.

5.2. **Hard Lefschetz theorem and mixed Hodge structures.** We want to state the hard Lefschetz theorem and the Hodge-Riemann bilinear relations in the language of Hodge structures. Let us recall briefly this formalism.

Hodge structures and polarizations.

Let $l \in \mathbb{Z}$, H be a finitely generated Abelian group, $H_{\mathbb{Q}} := H \otimes_{\mathbb{Z}} \mathbb{Q}$, $H_{\mathbb{R}} = H \otimes_{\mathbb{Z}} \mathbb{R}$, $H_{\mathbb{C}} = H \otimes_{\mathbb{Z}} \mathbb{C}$. A *pure Hodge structure of weight* l on H, $H_{\mathbb{Q}}$ or $H_{\mathbb{R}}$, is a direct sum decomposition $H_{\mathbb{C}} = \bigoplus_{p+q=l} H^{p,q}$ such that $H^{p,q} = \overline{H^{q,p}}$. The Hodge filtration is the decreasing filtration $F^p(H_{\mathbb{C}}) := \bigoplus_{p' \geq p} H^{p',q'}$. A *morphism of Hodge structures* $f : H \to H'$ is a group homomorphism such that the complexification of f (still denoted f) is compatible with the Hodge filtrations in the sense that $f(F^p H_{\mathbb{C}}) \subseteq F^p H'_{\mathbb{C}}$, i.e., such that it is a filtered map. Such maps are automatically what one calls *strict*, i.e., $(\operatorname{Im} f) \cap F^p H'_{\mathbb{C}} = f(F^p H_{\mathbb{C}})$. The category of Hodge structures of weight l with the above arrows is Abelian.

Let C be the *Weil operator*, i.e., the $\mathbb{R}$-linear map $C : H_{\mathbb{C}} \simeq H_{\mathbb{C}}$ such that $C(x) = i^{p-q}x$, for every $x \in H^{pq}$. Replacing i^{p-q} by $z^p \overline{z}^q$ we get a real action ρ of $\mathbb{C}^*$ on $H_{\mathbb{C}}$. A *polarization* of the real pure Hodge structure $H_{\mathbb{R}}$ is a real bilinear form Ψ on $H_{\mathbb{R}}$ which is invariant under the action given by ρ restricted to $S^1 \subseteq \mathbb{C}^*$ and such that the bilinear form $\widetilde{\Psi}(x,y) := \Psi(x, Cy)$ is symmetric and positive definite. If Ψ is a polarization, then Ψ is symmetric if l is even, and antisymmetric if l is odd. In any case, Ψ is nondegenerate. In addition, for every $0 \neq x \in H^{pq}$, $(-1)^l i^{p-q} \Psi(x, \overline{x}) > 0$, where Ψ also denotes the $\mathbb{C}$-bilinear extension of Ψ to $H_{\mathbb{C}}$.

Let η be the first Chern class of an ample line bundle on the projective n-fold Y. For every $r \geq 0$, define the space of *primitive vectors* $P^{n-r} := \operatorname{Ker} \eta^{r+1} \subseteq H^{n-r}(Y, \mathbb{Q})$.

Classical Hodge theory states that, for every l, $H^l(Y, \mathbb{Z})$ is a pure Hodge structure of weight l, P^{n-r} is a rational pure Hodge structure of weight $(n-r)$ polarized, up to a precise sign, by the bilinear form defined on $H^{n-r}(Y)$ as follows (it is well defined by Stokes' theorem):

$$S^{\eta}(\alpha, \beta) := \int_Y \eta^r \wedge \alpha \wedge \beta. \tag{37}$$

The fact that this form is nondegenerate is equivalent to the celebrated hard Lefschetz theorem. Its signature properties are expressed by the Hodge-Riemann bilinear relations.

Theorem 5.2.1. *Let Y be a complex projective manifold of dimension n. Then the following statements hold.*

(1) (**Hard Lefschetz theorem**) *For every $r \geq 0$ the cup product with η yields isomorphisms*

$$\eta^r : H^{n-r}(Y, \mathbb{Q}) \simeq H^{n+r}(Y, \mathbb{Q}).$$

(2) (**Primitive Lefschetz decomposition**) *For every $0 \leq r \leq n$ there is the direct sum decomposition*

$$H^{n-r}(Y, \mathbb{Q}) = \bigoplus_{j \geq 0} \eta^j P^{n-r-2j},$$

where each summand is a pure Hodge substructure of weight $n-r$ and all summands are mutually orthogonal with respect to the bilinear form S^{η}.

(3) (**Hodge-Riemann bilinear relations**) *For every $0 \leq r \leq n$, the bilinear form $(-1)^{\frac{(n-r)(n-r+1)}{2}} S^{\eta}$ is a polarization of the pure weight l Hodge*

structure $P^{n-r} \subseteq H^{n-r}(Y,\mathbb{R})$. In particular,

$$(38) \quad (-1)^{\frac{(n-r)(n-r-1)}{2}} i^{p-q} \int_Y \eta^r \wedge \alpha \wedge \overline{\alpha} \;>\; 0, \qquad \forall\; 0 \neq \alpha \in P^{n-r} \cap H^{p,q}(Y,\mathbb{C}).$$

Inductive approach to hard Lefschetz theorem.

Our proof (discussed in §3.3) of the decomposition theorem requires that we first establish the relative hard Lefschetz theorem. We do so by using an approach similar to the classical inductive approach to the hard Lefschetz Theorem 5.2.1(1). There are two variants of this inductive approach (see [62, 44]): the former is via Hodge-Riemann relations, the latter is via the semisimplicity of the monodromy action in a Lefschetz pencil. Though both are relevant to our approach to the decomposition theorem, we limit ourselves to discussing the former variant.

The induction is on $n := \dim Y$ and uses a nonsingular hyperplane section $D \subseteq Y$. The case $r = 0$ is trivial. The cases $r \geq 2$ follow by an easy induction on the dimension of Y using the Lefschetz hyperplane theorem. One is left with the key case $r = 1$. The cup product map $\eta := c_1(D) \wedge -$ factors as $\eta = g \circ r$:

$$H^{n-1}(Y) \xrightarrow{\;r\;} H^{n-1}(D) \xrightarrow{\;g\;} H^{n+1}(Y),$$

where r is the injective restriction map and g is the surjective Gysin map. It is easy to show that η is an isomorphism iff the intersection form on D, restricted to $\mathrm{Im}(r)$, is nondegenerate. While the form on $H^{n-1}(D)$ is nondegenerate by Poincaré duality, there is no a priori reason why it should restrict to a nondegenerate form on $H^{n-1}(Y)$. This is where the Hodge-Riemann relations enter the picture: by contradiction, assume that there is a nonzero class $\alpha \in \operatorname{Ker}\eta$, which we may suppose of pure Hodge type; then $r(\alpha)$ is primitive in $H^{n-1}(D)$, and, by the Hodge-Riemann relations on D, $0 = \int_Y \eta \wedge \alpha \wedge \overline{\alpha} = \int_D \alpha \wedge \overline{\alpha} \neq 0$, a contradiction.

The Lefschetz theorem on hyperplane sections coupled with the Hodge-Riemann bilinear relations for a hyperplane section imply the hard Lefschetz theorem for Y. However, they do not imply the Hodge-Riemann bilinear relations for the critical middle-dimensional cohomology group $H^n(Y)$, and the induction procedure grinds to a halt.

To make the proof work, one has to somehow establish the Hodge-Riemann relations on $H^n(Y)$. §3.3.2, sections 1 and 2b, outlines two instances of how Hodge-Riemann-type relations can be established.

The hard Lefschetz theorem applied to the fibers of a smooth projective morphism and Theorem 5.3.1 imply the following result (cf. item (3), following Theorem 1.6.1). For the proof, see [56] and [59], Théorème 4.2.6.

Theorem 5.2.2 (**Decomposition, semisimplicity and relative hard Lefschetz theorem for proper smooth maps).** *Let $f : X^n \to Y^m$ be a smooth proper map of smooth algebraic varieties of the indicated dimensions. Then*

$$f_*\mathbb{Q}_X \;\simeq\; \bigoplus_{j\geq 0} R^j f_*\mathbb{Q}_X[-j]$$

and the $R^j f_*\mathbb{Q}_X$ are semisimple local systems. If, in addition, f is projective and η is the first Chern class of an f-ample line bundle on X, then we have isomorphisms

$$\eta^r \;:\; R^{n-m-r} f_*\mathbb{Q}_X \;\simeq\; R^{n-m+r} f_*\mathbb{Q}_X,\ \forall r \geq 0,$$

and the local systems $R^j f_*\mathbb{Q}_X$ underlie polarizable variations of pure Hodge structures.

Mixed Hodge structures.

In general, the singular cohomology groups $H^j(Y,\mathbb{Z})$ of a singular variety cannot carry a pure Hodge structure of weight j; e.g., $H^1(\mathbb{C}^*,\mathbb{Z})$ has rank one, and pure Hodge structures of odd weight have even rank. However, these groups underlie a more subtle structure, the presence of which makes the topology of complex algebraic varieties even more remarkable.

Theorem 5.2.3 (**Mixed Hodge structure on cohomology).** *Let Y be an algebraic variety. For every $j \geq 0$ there is an increasing filtration (the weight filtration)*

$$\{0\} = W_{-1} \subseteq W_0 \subseteq \ldots \subseteq W_{2j} = H^j(Y,\mathbb{Q})$$

and a decreasing filtration (the Hodge filtration)

$$H^j(Y,\mathbb{C}) = F^0 \supseteq F^1 \supseteq \ldots \supseteq F^m \supseteq F^{m+1} = \{0\}$$

such that the filtrations induced by $F^\bullet$ on the complexified graded pieces of the weight filtration $W_\bullet$ endow every graded piece W_l/W_{l-1} with a rational pure Hodge structure of weight l. This structure (mixed Hodge structure) is functorial for maps of algebraic varieties and the induced maps strictly preserve both filtrations.

5.3. **The formalism of the constructible derived category.** Standard references for what follows are [116, 82, 19, 108, 9]; see also [162]. In what follows, we freely refer to our crash course in §1.5 and to the complete references given above.

A *full* subcategory $\mathcal{C}' \subseteq \mathcal{C}$ of a category $\mathcal{C}$ is a subcategory such that the induced map on the Hom sets is bijective; in other words we keep all the arrows.

An *additive* category $\mathcal{C}$ is one in which each $\mathrm{Hom}(A,B)$ is an Abelian group, composition of arrows is bilinear, the direct sum $A \oplus B$ is defined for any pair of objects $A, B \in \mathcal{C}$, and the zero object $0 \in \mathcal{C}$ exists. A *complex* K in an additive category $\mathcal{C}$ is a sequence

$$\cdots \longrightarrow K^{i-1} \xrightarrow{d^{i-1}} K^i \xrightarrow{d^i} K^{i+1} \xrightarrow{d^{i+1}} \cdots$$

of objects and morphisms in $\mathcal{C}$ such that for every $i \in \mathbb{Z}$ we have $d^i \circ d^{i-1} = 0$. The objects K^i are called the *entries* of the complex and the arrows d^i are called its *differentials*. One often omits the indexing of the arrows. A *map of complexes* $f : A \to B$ is a collection of arrows $f^i : A^i \to B^i$ such that $d \circ f = f \circ d$. Complexes in $\mathcal{C}$ form an additive category, denoted $C(\mathcal{C})$. Given a complex K and $m \in \mathbb{Z}$, the *m-shifted* complex $K[m]$ is the complex with entries $(K[m])^i := K^{i+m}$ and with differentials $d^i_{K[m]} = (-1)^k d^{i+k}_K$. The *cone* of a map of complexes $f : A \to B$ is the complex $Cone(f)$, where $Cone(f)^i := B^i \oplus A^{i+1}$ and the differential is defined by setting $d(b,a) = (d(b) + f(a), -d(a))$.

An *Abelian* category is an additive category where every arrow admits a kernel and a cokernel and, given any arrow $f : A \to B$, the resulting natural arrow $\mathrm{Coker}\,\{\mathrm{Ker} f \to A\} \to \mathrm{Ker}\,\{B \to \mathrm{Coker} f\}$ is an isomorphism.

In this paragraph, we work in a fixed Abelian category $\mathcal{A}$. An arrow is *monic* if its kernel is (isomorphic to) zero. If an arrow $A \to B$ is monic, then we say that A is a *subobject* of B. An object $A \in \mathcal{A}$ is *simple* if it has no nontrivial subobjects. The Abelian category $\mathcal{A}$ is *Artinian* if, for every object $A \in \mathcal{A}$, every descending chain of subobjects of A stabilizes. If $\mathcal{A}$ is Artinian, then every nonzero object A is a finite iterated extension of nonzero simple objects, called the *constituents* of A; the constituents of A are well defined up to isomorphism. The Abelian category

$\mathcal{A}$ is *Noetherian* if, for every object A, every ascending chain of subobjects of A stabilizes. The category of complexes $C(\mathcal{A})$ is Abelian. Given a complex K in $C(\mathcal{A})$ and an integer $i \in \mathbb{Z}$, we define the cohomology object $H^i(K) := \mathrm{Ker} d^i/\mathrm{Im} d^{i-1} \in \mathcal{A}$ and the truncated complexes $\tau_{\leq i}K$ and $\tau_{\geq i}K$ as follows:

$$(\tau_{\leq i}K)^l = K^l, \quad l < i, \qquad (\tau_{\leq i}K)^i = \mathrm{Ker}\, d^i, \qquad (\tau_{\leq i}K)^l = 0, \quad l > i,$$

with the obvious differentials, and

$$(\tau_{\geq i}K)^l = 0, \quad l < i, \qquad (\tau_{\leq i}K)^i = \mathrm{Coker}\, d^{i-1}, \qquad (\tau_{\leq i}K)^l = K^l, \quad l > i,$$

with the obvious differentials. For every $i \in \mathbb{Z}$ there are short exact sequences in the Abelian category $C(\mathcal{A})$:

$$0 \longrightarrow \tau_{\leq i}K \longrightarrow K \longrightarrow \tau_{\geq i+1}K \longrightarrow 0,$$

and natural identifications of functors

$$\tau_{\leq i} \circ \tau_{\geq i} \simeq \tau_{\geq i} \circ \tau_{\leq i} \simeq [-i] \circ H^i.$$

Given an arrow $f : A \to B$ in $C(\mathcal{A})$, we get a short exact sequence in $C(\mathcal{A})$:

$$0 \longrightarrow B \longrightarrow Cone(f : A \to B) \longrightarrow A[1] \longrightarrow 0.$$

Let Y be an algebraic variety and $\mathcal{D}_Y$ be the constructible bounded derived category (§1.5). The category $\mathcal{D}_Y$ is a *triangulated category*. In particular, it is additive, so that we can form finite direct sums, and it is equipped with the translation functor $A \mapsto A[1]$. A *triangle* is a diagram of maps $A \to B \to C \to A[1]$ in $\mathcal{D}_Y$. A most important feature of triangulated categories is the presence of distinguished triangles. Given a map of complexes $f : A' \to B'$, there exists a short exact sequence of complexes $0 \to B' \to Cone(f) \to A'[1] \to 0$. This exact sequence gives rise to a triangle $A' \to B' \to Cone(f) \to A'[1]$ in $\mathcal{D}_Y$. A *distinguished triangle* is a triangle which is isomorphic in $\mathcal{D}_Y$ to the one associated with a map f as above. Any map $f : A \to B$ in $\mathcal{D}_Y$ can be completed to a distinguished triangle. One should keep in mind that this construction is not functorial; see [82].

It is easy to show that the kernel of a morphism $f : A \to B$ in $\mathcal{D}_Y$ splits off as a direct summand of A. Since there are complexes which do not split nontrivially, *the category $\mathcal{D}_Y$ is not Abelian* (unless Y is a finite collection of points).

The cone construction is a replacement in the non-Abelian category $\mathcal{D}_Y$ of the notions of kernel and cokernel. In fact, if $f : A \to B$ is an injective (surjective, resp.) map of complexes, then $Cone(f)$ is isomorphic in $\mathcal{D}_Y$ to the cokernel (1-shifted kernel, resp.) complex.

An essential computational tool is that the application of a cohomological functor to a distinguished triangle produces a long exact sequence. Distinguished triangles are a replacement for short exact sequences in the non-Abelian category $\mathcal{D}_Y$. A *cohomological functor*, with values in an Abelian category $\mathcal{A}$, is an additive functor $T : \mathcal{D}_Y \to \mathcal{A}$ such that $T(A) \to T(B) \to T(C)$ is exact for every distinguished triangle as above. Setting $T^i(A) := T(A[i])$, we get the long exact sequence

$$\cdots \longrightarrow T^i(A) \longrightarrow T^i(B) \longrightarrow T^i(C) \longrightarrow T^{i+1}(A) \longrightarrow \cdots.$$

Using injective resolutions and the two global sections functors Γ and Γ_c we define the *derived global sections functors* (see [82, 116] for the identification with categorical derived functors)

$$R\Gamma, R\Gamma_c \ : \ \mathcal{D}_Y \longrightarrow \mathcal{D}_{pt},$$

and the finite-dimensional cohomology vector spaces of Y with coefficients in $K \in \mathcal{D}_Y$:

$$H^*(Y, K) := H^*(R\Gamma(Y, K)), \qquad H^*_c(Y, K) := H^*(R\Gamma_c(Y, K)) \text{ (compact supports)}.$$

Given a map $f : X \to Y$, we have the *four functors*

$$Rf_*, Rf_! \,:\, \mathcal{D}_X \longrightarrow \mathcal{D}_Y, \qquad f^*, f^! \,:\, \mathcal{D}_Y \longrightarrow \mathcal{D}_X.$$

The sheaf-theoretic direct image functors $f_*, f_! : Sh_X \to Sh_Y$ are *left exact* as functors, e.g., if $0 \to F \to G \to H \to 0$ is an exact sequence of sheaves on X, then $0 \to f_!F \to f_!G \to f_!H$ is an exact sequence of sheaves on Y. The *right derived* functors Rf_* and $Rf_!$ arise by applying the sheaf-theoretic direct image functors f_* and $f_!$ (proper supports), term-by-term, to injective resolutions. Taking cohomology sheaves, we obtain the i-th right derived functors R^if_* and $R^if_!$. We have equalities of sheaf-theoretic functors $R^0f_* = f_*$, $R^0f_! = f_!$. The *inverse image* functor $f^* : Sh_Y \to Sh_X$ is exact on sheaves and descends to the derived category. The *exceptional inverse image functor* $f^!$ does not arise from a functor defined on sheaves.

It is customary to employ the following simplified notation to denote the four functors $(f^*, f_*, f_!, f^!)$. In this paper, f_* and $f_!$ denote the right derived functors. To avoid confusion, the sheaf-theoretic functors are denoted R^0f_*, $R^0f_!$.

Given maps $f : X \to Y$, $g : Y \to Z$, we have $(g \circ f)^! = f^! \circ g^!$, etc. For $g : Y \to pt$ and for $C \in \mathcal{D}_X$, we have canonical isomorphisms

$$H^*(X, C) \simeq H^*(Y, f_*C), \qquad H^*_c(X, C) \simeq H^*(Y, f_!C).$$

The functors $f_!$ and $f^!$ in special cases.
If f is proper, e.g., a closed immersion, then $f_! = f_*$.
If f is smooth of relative dimension d, then $f^! = f^*[2d]$.

A closed embedding $f : X \to Y$ is *normally nonsingular of pure codimension d* ([87]) if it can be realized as the intersection $X = Y \cap N$ inside M, where N, M are nonsingular, N has codimension d in M, and N is transverse to every stratum of some stratification Σ of Y. In this case, we have that $f^! = f^*[-2d]$ holds for every Σ-constructible complex. Such so-called normally nonsingular inclusions can be obtained by embedding Y in some projective space and then intersecting Y with d general hypersurfaces.
If f is an open embedding, then $f^! = f^*$.
If f is a locally closed embedding, then
1) $f_!$ is the extension-by-zero functor and $f_! = R^0f_!$;
2) $f^! = f^*R\Gamma_X$, where Γ_XF, not to be confused with the sheaf $f_!f^*F = F_X$ that is zero outside X and coincides with F on X, is the sheaf of sections of the sheaf F supported on X (see [116], p.95). If, in addition, f is a closed embedding, then $H^*(Y, f_!f^!K) = H^*(X, f^!K) = H^*(Y, Y \setminus X; K) = H^*_X(Y, K)$.

The usual Hom complex construction can be derived, and we get the right derived functors

$$\mathrm{RHom} : \mathcal{D}_Y^{opp} \times \mathcal{D}_Y \longrightarrow \mathcal{D}_{pt}, \quad R\mathcal{H}om : \mathcal{D}_Y^{opp} \times \mathcal{D}_Y \longrightarrow \mathcal{D}_Y$$

with the associated Ext^i and $\mathcal{E}xt^i$ functors. We have

$$\mathrm{Hom}_{\mathcal{D}_Y}(K, K') = H^0(Y, R\mathcal{H}om(K, K')) = H^0(Y, \mathrm{RHom}(K, K')).$$

The pair (f^*, f_*) is an *adjoint pair* (this holds also for the sheaf-theoretic version) and so is $(f_!, f^!)$ and we have, for every $C \in \mathcal{D}_X$ and $K \in \mathcal{D}_Y$:

$$f_* R\mathcal{H}om(f^*K, C) = R\mathcal{H}om(K, f_*C), \qquad f_* R\mathcal{H}om(f_!C, K) = R\mathcal{H}om(C, f^!K).$$

Since we are working with field coefficients, the tensor product operation $\otimes$ on complexes is exact and there is no need to derive it. For $K_i \in \mathcal{D}_Y$, we have (also for $R\mathcal{H}om$):

$$\mathrm{RHom}(K_1 \otimes K_2, K_3) = \mathrm{RHom}(K_1, R\mathcal{H}om(K_2, K_3))$$

and, if the sheaves $\mathcal{H}^i(K_3)$ are locally constant:

$$R\mathcal{H}om(K_1, K_2 \otimes K_3) = R\mathcal{H}om(K_1, K_2) \otimes K_3.$$

The *dualizing complex* $\omega_Y \in \mathcal{D}_Y$ is well defined up to canonical isomorphism by setting $\omega_Y := \gamma^! \mathbb{Q}_{pt}$, where $\gamma : Y \to pt$. If Y is nonsingular, then $\omega_Y \simeq \mathbb{Q}_Y[2\dim_{\mathbb{C}} Y]$. Given $f : X \to Y$, we have $\omega_X = f^! \omega_Y$. Define a contravariant functor

$$D \,:\, \mathcal{D}_Y \longrightarrow \mathcal{D}_Y, \qquad K \longmapsto D(K)\,(= K^\vee) \,:=\, R\mathcal{H}om(K, \omega_Y).$$

We have $D^2 = \mathrm{Id}$, $(K[i])^\vee = K^\vee[-i]$ and $\omega_Y = \mathbb{Q}_Y^\vee$. The complex $K^\vee$ is called the (Verdier) *dual* of K. Poincaré-Verdier duality consists of the canonical isomorphism

$$H^i(Y, K^\vee) \simeq H_c^{-i}(Y, K)^\vee,$$

which is a formal consequence of the fact that $(f_!, f^!)$ form an adjoint pair. The usual Poincaré duality for topological manifolds is the special case when Y is smooth and orientable, for then a choice of orientation gives a natural isomorphism $\omega_Y \simeq \mathbb{Q}_Y[\dim_{\mathbb{R}} Y]$.

We have the important relations

$$Df_! = f_*D, \qquad Df^! = f^*D.$$

A *t-structure* on a triangulated category $\mathcal{D}$ is the data of two full subcategories $\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0} \subseteq \mathcal{D}$ subject to the following three requirements:

(1) for every $C \in \mathcal{D}^{\leq 0}$ and $C' \in \mathcal{D}^{\geq 1}$, we have $\mathrm{Hom}_{\mathcal{D}}(C, C') = 0$;
(2) $D^{\leq 0}[1] \subseteq D^{\leq 0}$ and $D^{\geq 0} \subseteq D^{\geq 0}[1]$;
(3) for every $C \in \mathcal{D}$, there is a distinguished triangle $C' \to C \to C'' \to C'[1]$ with $C' \in D^{\leq 0}$ and $C'' \in D^{\geq 1}$.

A *t-category* is a triangulated category endowed with a t-structure ([9, 116]). The heart of a t-structure is the full subcategory $\mathcal{C} := \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$. The heart of a t-structure is an Abelian category. By virtue of axiom (1), the distinguished triangle in (3) is defined up to canonical isomorphism and this defines functors, called the truncation functors $\tau_{\leq 0} : \mathcal{D} \to \mathcal{D}^{\leq 0}$, $C \mapsto C' =: \tau_{\leq 0} C$ and $\tau_{\geq 0} : \mathcal{D} \to \mathcal{D}^{\geq 0}$, $C \mapsto (C[-1])''[1] =: \tau_{\geq 0} C$. The functor $H^0 := \tau_{\leq 0} \circ \tau_{\geq 0} : \mathcal{D} \to \mathcal{C}$ is cohomological.

The prototype of a t-structure is the *standard t-structure* on $\mathcal{D}_Y$, which is defined by setting $\mathcal{D}_Y^{\leq 0} \subseteq \mathcal{D}_Y$ to be the full subcategory of complexes $K \in \mathcal{D}_Y$ with $\mathcal{H}^j(K) = 0$ for $j > 0$, and $\mathcal{D}_Y^{\geq 0} \subseteq \mathcal{D}_Y$ to be the full subcategory of complexes $K \in \mathcal{D}_Y$ with $\mathcal{H}^j(K) = 0$ for $j < 0$. The three axioms are easily verified. The truncation functors are the usual ones. The intersection $\mathcal{D}_Y^{\leq 0} \cap \mathcal{D}_Y^{\geq 0}$ is the Abelian category of constructible sheaves on Y. The two-sided truncation $\tau_{\leq 0} \circ \tau_{\geq 0}$ is the usual functor $\mathcal{H}^0$ (0^{th}-cohomology sheaf).

Another important t-structure is the (middle) *perverse* t-structure (§2.3).

We have the following notions of exactness. A functor of Abelian categories is *exact* if it preserves exact sequences. We also have the companion notions of *left and right exactness*. A *functor of triangulated categories* (i.e., additive an commuting with translations) is *exact* if it preserves distinguished triangles. A *functor of t-categories* $F : \mathcal{D} \to \mathcal{D}'$ is a functor of the underlying triangulated categories. It is *exact* if it preserves distinguished triangles. It is *left t-exact* if $F : \mathcal{D}^{\geq 0} \to \mathcal{D}'^{\geq 0}$. It is *right t-exact* if $F : \mathcal{D}^{\leq 0} \to \mathcal{D}'^{\leq 0}$. It is *$t$-exact* if it is both left and right t-exact, in which case it preserves the Abelian hearts; i.e., it induces an exact functor $F : \mathcal{C} \to \mathcal{C}'$ of Abelian categories.

Perverse t-exactness.
Let $f : X \to Y$ be a map of varieties. If $\dim f^{-1}y \leq d$, then

$$f_!, f^* : {}^{\mathfrak{p}}\mathcal{D}_Y^{\leq 0} \longrightarrow {}^{\mathfrak{p}}\mathcal{D}_Y^{\leq d}, \qquad f^!, f_* : {}^{\mathfrak{p}}\mathcal{D}_Y^{\geq 0} \longrightarrow {}^{\mathfrak{p}}\mathcal{D}_Y^{\geq -d}.$$

If f is quasi-finite (= finite fibers), then $d = 0$ above.
If f is affine, e.g., the embedding of the complement of a Cartier divisor, the embedding of an affine open subset, or the projection of the complement of a universal hyperplane section, etc., then

$$f_* : {}^{\mathfrak{p}}\mathcal{D}_Y^{\leq 0} \longrightarrow {}^{\mathfrak{p}}\mathcal{D}_Y^{\leq 0} \quad \text{(right } t\text{-exact)}, \qquad f_! : {}^{\mathfrak{p}}\mathcal{D}_Y^{\geq 0} \longrightarrow {}^{\mathfrak{p}}\mathcal{D}_Y^{\geq 0} \quad \text{(left } t\text{-exact)}.$$

More generally, if locally over Y, X is the union of $d+1$ affine open sets, then

$$f_* : {}^{\mathfrak{p}}\mathcal{D}_Y^{\leq 0} \longrightarrow {}^{\mathfrak{p}}\mathcal{D}_Y^{\leq d}, \qquad f_! : {}^{\mathfrak{p}}\mathcal{D}_Y^{\geq 0} \longrightarrow {}^{\mathfrak{p}}\mathcal{D}_Y^{\geq -d}.$$

If f is quasi-finite and affine, then $f_!$ and f_* are t-exact.
If f is finite (= proper and finite fibers), then $f_! = f_*$ are t-exact.
If f is a closed embedding, then $f_! = f_*$ are t-exact and fully faithful. In this case it is customary to drop f_* from the notation, e.g., $IC_X \in \mathcal{D}_Y$.
If f is smooth of relative dimension d, then $f^![-d] = f^*[d]$ are t-exact.
In particular, if f is étale, then $f^! = f^*$ are t-exact.
If f is a normally nonsingular inclusion of codimension d with respect to a stratification Σ of Y, then $f^![d] = f^*[-d] : \mathcal{D}_Y^{\Sigma} \to \mathcal{D}_X$ are t-exact.

The following splitting criterion ([56, 58]) plays an important role in the proof of the decomposition theorem:

Theorem 5.3.1. *Let $K \in \mathcal{D}_X$ and $\eta : K \to K[2]$ such that $\eta^l : {}^{\mathfrak{p}}\mathcal{H}^{-l}(K) \to {}^{\mathfrak{p}}\mathcal{H}^{l}(K)$ is an isomorphism for all l. Then there is an isomorphism in $\mathcal{D}_Y$:*

$$K \simeq \bigoplus_i {}^{\mathfrak{p}}\mathcal{H}^i(K)[-i].$$

5.4. **Familiar objects from algebraic topology.** Here is a brief list of some of the basic objects of algebraic topology and a short discussion of how they relate to the formalism in $\mathcal{D}_Y$.

(Co)homology, etc.:
singular cohomology: $H^l(Y, \mathbb{Q}_Y)$;
singular cohomology with compact supports: $H^l_c(Y, \mathbb{Q}_Y)$;
singular homology $H_l(Y, \mathbb{Q}) = H_c^{-l}(Y, \omega_Y)$;
Borel-Moore homology: $H_l^{BM}(Y, \mathbb{Q}) = H^{-l}(Y, \omega_Y)$;
relative (co)homology: if $i : Z \to Y$ is a locally closed embedding and $j : (Y \setminus Z) \to Y$, then we have canonical isomorphisms $H^l(Y, Z, \mathbb{Q}) \simeq H^l(Y, i_! i^! \mathbb{Q})$ and $H_l(Y, Z, \mathbb{Q}) \simeq H_c^{-l}(Y, j_* j^* \omega_Y)$.

Intersection (co)homology. The intersection homology groups $IH_j(Y)$ of an n-dimensional irreducible variety Y are defined as the j-th homology groups of chain complexes of geometric chains with closed supports subject to certain admissibility conditions ([86]). Similarly, one defines intersection homology with compact supports from the natural maps

$$IH_j(Y) \longrightarrow H_j^{BM}(Y), \qquad IH_{c,j}(Y) \longrightarrow H_j(Y).$$

intersection cohomology: $IH^j(Y) := IH_{2n-j}(Y) = H^{-n+j}(Y, IC_Y)$.
intersection cohomology with compact supports: $IH_c^j(Y) := IH_{c,2n-j}(Y) = H_c^{-n+j}(Y, IC_Y)$.

Duality and pairings. Verdier duality implies we have canonical identifications

$$H_l(Y,\mathbb{Q})^\vee = H_c^{-l}(Y,\omega_Y)^\vee \simeq H^l(Y,\mathbb{Q}),\ H_l^{BM}(Y,\mathbb{Q})^\vee = H^{-l}(Y,\omega_Y)^\vee \simeq H_c^l(Y,\mathbb{Q}).$$

If Y is nonsingular of dimension n, then we have the *Poincaré duality* isomorphisms:

$$H^{n+l}(Y,\mathbb{Q}) \simeq H_{n-l}^{BM}(Y,\mathbb{Q}), \qquad H_{n+l}(Y,\mathbb{Q}) \simeq H_c^{n-l}(Y,\mathbb{Q}).$$

There are two ways to express the classical nondegenerate Poincaré intersection pairing:

$$H^{n+l}(Y,\mathbb{Q}) \times H_c^{n-l}(Y,\mathbb{Q}) \longrightarrow \mathbb{Q}, \qquad H_{n-l}^{BM}(Y,\mathbb{Q}) \times H_{n+l}(Y,\mathbb{Q}) \longrightarrow \mathbb{Q}.$$

While the former one is given by wedge product and integration, the latter can be described geometrically as the intersection form in Y as follows. Given a Borel-Moore cycle and a usual, i.e., compact, cycle in complementary dimensions, one changes one of them, say the first one, to one homologous to it, but transverse to the other. Since the ordinary one has compact supports, the intersection set is finite and one gets a finite intersection index.

Let Y be compact, Z be a closed subvariety such that $Y \setminus Z$ is smooth and of pure dimension n. We have the *Lefschetz duality*

$$\begin{aligned} H_q(Y,Z;\mathbb{Q}) &= H_c^{-q}(Y, j_*j^*\omega_Y) = H^{-q}(Y, j_*j^*\omega_Y) \\ &= H^{-q}(Y\setminus Z, \mathbb{Q}_Y[2n]) = H^{2n-q}(Y\setminus Z,\mathbb{Q}). \end{aligned}$$

Goresky-MacPherson's Poincaré duality: since $IC_Y \simeq IC_Y^\vee$, we have canonical isomorphisms

$$IH^{n+l}(Y,\mathbb{Q}) \simeq IH_c^{n-l}(Y,\mathbb{Q})^\vee.$$

Functoriality. The usual maps in (co)homology associated with a map $f : X \to Y$ arise from the adjunction maps

$$\mathbb{Q}_Y \longrightarrow f_*f^*\mathbb{Q}_Y = f_*\mathbb{Q}_X, \qquad f_!f^!\omega_Y = f_!\omega_X \longrightarrow \omega_Y$$

by taking cohomology. In general, for an arbitrary map f, there are no maps associated with Borel-Moore and cohomology with compact supports. If f is proper, then $f_* = f_!$ and one gets pull-back for proper maps in cohomology with compact supports and push-forward for proper maps in Borel-Moore homology. These maps are dual to each other.

If f is an open immersion, then $f^* = f^!$ and one has the restriction to an open subset map for Borel-Moore homology and the push-forward for an open subset map for cohomology with compact supports. These maps are dual to each other.

Cup and Cap products. The natural isomorphisms $H^l(Y,\mathbb{Q}) \simeq \mathrm{Hom}_{\mathcal{D}_Y}(\mathbb{Q}_Y, \mathbb{Q}_Y[l])$ and $\mathrm{Hom}_{\mathcal{D}_Y}(\mathbb{Q}_Y, \mathbb{Q}_Y[l]) \simeq \mathrm{Hom}_{\mathcal{D}_Y}(\mathbb{Q}_Y[k], \mathbb{Q}_Y[k+l])$ identify the *cup product*

$$\cup : H^l(Y,\mathbb{Q}) \times H^k(Y,\mathbb{Q}) \to H^{k+l}(Y,\mathbb{Q})$$

with the composition

$$\mathrm{Hom}_{\mathcal{D}_Y}(\mathbb{Q}_Y,\mathbb{Q}_Y[l]) \times \mathrm{Hom}_{\mathcal{D}_Y}(\mathbb{Q}_Y[l],\mathbb{Q}_Y[k+l]) \longrightarrow \mathrm{Hom}_{\mathcal{D}_Y}(\mathbb{Q}_Y,\mathbb{Q}_Y[k+l]).$$

Similarly, the *cap product*

$$\cap : H_k^{BM}(Y,\mathbb{Q}) \times H^l(Y, Y\setminus Z,\mathbb{Q}) \longrightarrow H_{k-l}^{BM}(Z,\mathbb{Q})$$

relative to a closed imbedding $i : Z \to Y$ is obtained as a composition of maps in the derived category as follows:

$$\begin{array}{ccc}
H_k^{BM}(Y,\mathbb{Q}) & & H^l(Y,Y\setminus Z,\mathbb{Q}) = \mathrm{Hom}_{\mathcal{D}_Z}(\mathbb{Q}_Z, i^!\mathbb{Q}_Y[l]) \\
\| & & \times \\
\mathrm{Hom}_{\mathcal{D}_Y}(\mathbb{Q}_Y,\omega_Y[-k]) & \longrightarrow & \mathrm{Hom}_{\mathcal{D}_Z}(i^!\mathbb{Q}_Y, i^!\omega_Y[-k]) = \mathrm{Hom}_{\mathcal{D}_Z}(i^!\mathbb{Q}_Y,\omega_Z[-k]) \\
 & & \downarrow \\
 & & H_{k-l}^{BM}(Z,\mathbb{Q}) = \mathrm{Hom}_{\mathcal{D}_Z}(\mathbb{Q}_Z,\omega_Z[l-k]).
\end{array}$$

Gysin map. Let $i : Z \to Y$ be the closed embedding of a codimension d complex submanifold of the complex manifold Y. We have $i_* = i_!$ and $i^! = i^*[-2d]$, the adjunction map for $i_!$ yields

$$i_*\mathbb{Q}_Z = i_!i^*\mathbb{Q}_Y = i_!i^!\mathbb{Q}_Y[2d] \longrightarrow \mathbb{Q}_Y[2d]$$

and by taking cohomology we get the Gysin map

$$H^l(Z,\mathbb{Q}) \longrightarrow H^{l+2d}(Y,\mathbb{Q}).$$

Geometrically, this can be viewed as equivalent via Poincaré duality to the proper push-forward map in Borel-Moore homology $H_j^{BM}(Z,\mathbb{Q}) \to H_j^{BM}(Y,\mathbb{Q})$.

Fundamental class. Let $i : Z \to Y$ be the closed immersion of a d-dimensional subvariety of the manifold Y. The space Z carries a fundamental class in $H_{2d}^{BM}(Z)$. The fundamental class of Z is the image of this class in $H_{2d}^{BM}(Y) \simeq H^{2n-2d}(Y,\mathbb{Z})$.

Mayer-Vietoris. There is a whole host of Mayer-Vietoris sequences (cf. [116], 2.6.10), e.g.:

$$\cdots \to H^{l-1}(U_1\cap U_2, K) \to H^l(U_1 \cup U_2, K) \to H^l(U_1,K)\oplus H^l(U_2,K) \to \cdots .$$

Relative (co)homology. Let $U \xrightarrow{j} Y \xleftarrow{i} Z$ be the inclusions of an open subset $U \subset Y$ and of the closed complement $Z := Y\setminus U$. We have the following "attaching" distinguished triangles:

$$i_!i^!C \longrightarrow C \longrightarrow j_*j^*C \xrightarrow{[1]}, \qquad j_!j^!C \longrightarrow C \longrightarrow i_*i^*C \xrightarrow{[1]} .$$

The long exact sequences of relative (co)homology (including the versions with compact supports) arise by taking the associated long exact sequences.

Refined intersection forms. Let $i : Z \to Y$ be a closed immersion into a nonsingular variety Y of dimension n. There are maps

$$i_!\omega_Z[-n] = i_!i^!\omega_Y[-n] \longrightarrow \omega_Y[-n] \simeq \mathbb{Q}_Y[n] \longrightarrow i_*i^*\mathbb{Q}_Y[n] = i_*\mathbb{Q}_Z[n].$$

Taking cohomology we get the *refined intersection form* on $Z \subseteq Y$, which we can view in two equivalent ways as a linear or a bilinear map:

$$H_{n-l}^{BM}(Z) \longrightarrow H^{n+l}(Z) \quad \text{or} \qquad H_{n-l}^{BM}(Z) \times H_{n+l}(Z) \longrightarrow \mathbb{Q}.$$

It is called refined because we are intersecting cycles in the nonsingular Y which are supported on Z. By using Lefschetz duality, this pairing can be viewed as the cup product in relative cohomology. These forms play an important role in our proof of the decomposition theorem [48, 51] (see §3.3).

5.5. **Nearby and vanishing cycle functors.** An important feature of perverse sheaves is their stability for the two functors Ψ_f, Φ_f. These functors were defined in [57] in the context of étale cohomology as a generalization of the notion of vanishing cycle in the classical Picard-Lefschetz theory. As explained in §5.7.2, they play a major role in the description of the possible extensions of a perverse sheaf through a principal divisor. We discuss these functors in the complex analytic setting. Let $f : X \to \mathbb{C}$ be a regular function and $X_0 \subseteq X$ be its divisor, that is, $X_0 = f^{-1}(0)$. We are going to define functors $\Psi_f, \Phi_f : \mathcal{D}_X \to \mathcal{D}_{X_0}$ which send perverse sheaves on X to perverse sheaves on X_0. We follow the convention for shifts employed in [116].

Let $e : \mathbb{C} \to \mathbb{C}$ be the map $e(\zeta) = \exp(2\pi\sqrt{-1}\zeta)$ and consider the following diagram:

$$\begin{tikzcd} X_\infty := X \times_e \mathbb{C} \arrow[r] \arrow[d] \arrow[rr, bend left, "p"] & X^* \arrow[r, "j"] \arrow[d] & X \arrow[d] & X_0 \arrow[l, "i"'] \arrow[d] \\ \mathbb{C} \arrow[r] \arrow[rr, bend right, "e"'] & \mathbb{C}^* \arrow[r] & \mathbb{C} & \{o\} \arrow[l] \end{tikzcd}$$

For $K \in \mathcal{D}_X$, the nearby cycle functor $\Psi_f(K) \in \mathcal{D}_{X_0}$ is defined as:

$$\Psi_f(K) := i^* p_* p^* K.$$

Note that $\Psi_f(K)$ depends only on the restriction of K to X^*. It can be shown that $\Psi_f(K)$ is constructible. Depending on the context, we shall consider Ψ_f as a functor defined on $\mathcal{D}_X$, or on $\mathcal{D}_{X^*}$.

The group $\mathbb{Z}$ of deck transformations $\zeta \to \zeta + n$ acts on X_∞ and therefore on $\Psi_f(K)$. We denote by $T : \Psi_f(K) \to \Psi_f(K)$ the positive generator of this action.

Remark 5.5.1. (See [85], §6.13 for details.) Under mild hypotheses, for instance if f is proper, there exists a continuous map $r : U \to X_0$ of a neighborhood of X_0, compatible with the stratification, whose restriction to X_0 is homotopic to the identity map. Denote by r_ϵ the restriction of r to $f^{-1}(\epsilon)$, with $\epsilon \in \mathbb{C}$ small enough so that $f^{-1}(\epsilon) \subseteq U$. Let $X_\epsilon := f^{-1}(\epsilon)$. Then

$$r_{\epsilon *}(K_{|X_\epsilon}) = \Psi_f(K).$$

In particular, let $x_0 \in X_0$, let N be a neighborhood of x_0 contained in U and let $\epsilon \in \mathbb{C}$ be as before. Then the cohomology sheaves of $\Psi_f(K)$ can be described as follows:

$$\mathcal{H}^i(\Psi_f(K))_{x_0} = H^i(N \cap f^{-1}(\epsilon), K_{|N \cap f^{-1}(\epsilon)}).$$

The monodromy $X_\epsilon \to X_\epsilon$ induces a transformation $T : \Psi_f(K) \to \Psi_f(K)$ called the *monodromy transformation.*

Example 5.5.2. Let $X = \mathbb{C}$ and K be a local system on $\mathbb{C}^*$. Since the inverse image by e of a disk centered at 0 is contractible, $\Psi_f(K)$ can be identified with the stalk at some base point x_0. The automorphism T is just the monodromy of the local system.

The adjunction map $K \to p_*p^*K$ gives a natural morphism $i^*K \to \Psi_f(K)$. The vanishing cycle complex $\Phi_f(K) \in \mathcal{D}_{X_0}$ fits in the following distinguished triangle:

$$i^*K \longrightarrow \Psi_f(K) \xrightarrow{\text{can}} \Phi_f(K)[1] \xrightarrow{[1]} . \tag{39}$$

This distinguished triangle determines $\Phi_f(K)$ only up to a nonunique isomorphism. The definition of Φ_f as a functor requires more care; see [116]. The long exact sequence for the cohomology sheaves of this distinguished triangle and Remark 5.5.1 show that

$$\mathcal{H}^i(\Phi_f(K))_{x_0} = H^i(N, N \cap f^{-1}(\epsilon), K).$$

Just as the nearby cycle functor, the vanishing cycle $\Phi_f(K)$ is endowed with an automorphism T.

We now list some of the properties of the functors Ψ_f and Φ_f:

Theorem 5.5.3.

(1) *The functors commute, up to a shift, with Verdier duality (see* [104]*, and* [30]*):*

$$\Psi_f(DK) = D\Psi_f(K)[2] \qquad \Phi_f(DK) = D\Phi_f(K)[2].$$

(2) *If K is a perverse sheaf on X, then $\Psi_f(K)[-1]$ and $\Phi_f(K)[-1]$ are perverse sheaves on X_0 (see* [85] *6.13,* [9], [30], [104]*).*

(3) *Dualizing the distinguished triangle (*39*) we get a distinguished triangle*

$$i^!K \longrightarrow \Phi_f(K) \xrightarrow{\text{var}} \Psi_f(K)[-1] \xrightarrow{[1]} , \tag{40}$$

with the property that

$$\text{can} \circ \text{var} = T - I : \Phi_f(K) \to \Phi_f(K), \qquad \text{var} \circ \text{can} = T - I : \Psi_f(K) \to \Psi_f(K),$$

and we have the fundamental octahedron of complexes of sheaves on X_0 :

$$\begin{tikzcd}
 & i^*j_*j^*K \arrow[ddl] \arrow[dr] & \\
i^*K[1] \arrow[ur] \arrow[d, "{[1]}"'] & & i^!K[1] \arrow[ll, dashed] \arrow[ddl, dashed] \\
\Psi_f(K) \arrow[rr, "T-I"] \arrow[dr, "\text{can}"'] & & \Psi_f(K) \arrow[u] \arrow[uul, "{[1]}"'] \\
 & \Phi_f(K)[1] \arrow[uul, dashed] \arrow[ur, "\text{var}[1]"'] &
\end{tikzcd}$$

Remark 5.5.4. Clearly, if $U \subseteq X$ is an open subset, then the restriction to U of $\Psi_f(K)$ is the nearby cycle complex of the restriction $K_{|U}$ relative to the function $f_{|U}$ for $X \cap U$. On the other hand, explicit examples show that $\Psi_f(K)$ depends on f and not only on the divisor X_0: the nearby functors associated with different defining equations of X_0 may differ. In particular, it is not possible to define the functor Ψ_f if the divisor X_0 is only locally principal. Verdier has proposed in [171] an alternative functor, which he called the "specialization functor" $\mathrm{Sp}_{Y,X} : \mathcal{D}_X \to \mathcal{D}_{C_Y}$, associated with any closed imbedding $Y \to X$, where C_Y is the normal cone of Y in X. In the particular case that Y is a locally principal divisor in X, the specialization functor is related to the nearby functor as follows: the normal cone C_Y is a line bundle, and a local defining equation f of Y defined on an open set $V \subseteq X$ defines a section $s_f : Y \cap V \to C_{Y \cap V}$ trivializing the fibration. One has an isomorphism of functors $s_f^* \mathrm{Sp}_{Y,X} \simeq \Psi_f$.

5.6. **Unipotent nearby and vanishing cycle functors.** Let K be a perverse sheaf on $X \setminus X_0$. The map $j : X \setminus X_0 \to X$ is affine, so that j_*K and $j_!K$ are perverse sheaves on X.

Let us consider the ascending chain of perverse subsheaves

$$\mathrm{Ker}\,\{(T-I)^N : \Psi_f(K)[-1] \to \Psi_f(K)[-1]\}.$$

For $N \gg 0$ this sequence stabilizes because of the Noetherian property of the category of perverse sheaves. We call the resulting T-invariant perverse subsheaf the unipotent nearby cycle perverse sheaf associated with K and we denote it by $\Psi_f^u(K)$. In exactly the same way, it is possible to define the unipotent vanishing cycle functor Φ_f^u :

$$\Phi_f^u(K) = \mathrm{Ker}\,\{\,(T-I)^N : \Phi_f(K) \to \Phi_f(K)\,\}, \qquad \text{for } N \gg 0.$$

The perverse sheaves $\Psi_f(K)[-1]$ and $\Phi_f(K)[-1]$ are in fact the direct sum of Ψ_f^u and another T-invariant subsheaf on which $(T-I)$ is invertible.

Remark 5.6.1. The functor $\Psi_f(K)$ on a perverse sheaf K can be reconstructed from Ψ_f^u by applying this latter to the twists of K with the pull-back by f of local systems on $\mathbb{C}^*$; see [7], p.47.

We have the useful formulæ

$$\mathrm{Ker}\,\{\,j_!K \to j_{!*}K\,\} \simeq \mathrm{Ker}\,\{\,\Psi_f^u(K) \xrightarrow{T-I} \Psi_f^u(K)\,\},$$
$$\mathrm{Coker}\,\{\,j_{!*}(K) \to j_*K\,\} \simeq \mathrm{Coker}\,\{\,\Psi_f^u(K) \xrightarrow{T-I} \Psi_f^u(K)\,\}.$$

They can be derived as follows. The cone of $(T-I) : \Psi_f(K) \to \Psi_f(K)$, which is isomorphic to i^*j_*K, is also isomorphic, up to a shift [1], to the cone of $(T-I) : \Psi_f^u(K) \to \Psi_f^u(K)$, and we still have the distinguished triangle

$$i^*j_*K \xrightarrow{[1]} \Psi_f^u(K) \xrightarrow{T-I} \Psi_f^u(K) \longrightarrow \ .$$

The long exact sequence of perverse cohomology introduced in §2.5 then gives

$${}^{\mathfrak{p}}\mathcal{H}^{-1}(i^*j_*K) = \mathrm{Ker}\,\{\,\Psi_f^u(K) \xrightarrow{T-I} \Psi_f^u(K)\,\}$$

and

$${}^{\mathfrak{p}}\mathcal{H}^{0}(i^*j_*K) = \mathrm{Coker}\,\{\,\Psi_f^u(K) \xrightarrow{T-I} \Psi_f^u(K)\,\}.$$

In turn, the long exact perverse cohomology sequence of the distinguished triangle

$$i^*j_*K \xrightarrow{[1]} j_!K \longrightarrow j_*K \longrightarrow$$

and the fact that j_*K and $j_!K$ are perverse sheaves on X give

$${}^{\mathfrak{p}}\mathcal{H}^{-1}(i^*j_*K) = \mathrm{Ker}\,\{\,j_!K \to j_*K\,\} = \mathrm{Ker}\,\{\,j_!K \to j_{!*}K\}$$

and

$$^{\mathfrak{p}}\mathcal{H}^0(i^*j_*K) = \mathrm{Coker}\,\{\,j_!K \to j_*K\,\} = \mathrm{Coker}\,\{\,j_{!*}K \to j_*K\,\}.$$

Remark 5.6.2. Let N be a nilpotent endomorphism of an object M of an abelian category. Suppose $N^{k+1}=0$. By [62], 1.6, there exists a unique finite increasing filtration

$$M_\bullet : \{0\} \subseteq M_{-k} \subseteq \ldots \subseteq M_k = M$$

such that

$$NM_l \subseteq M_{l-2} \ \text{ and } \ N^l : M_l/M_{l-1} \simeq M_{-l}/M_{-l-1}.$$

The filtration defined in this way by $T-I$ on $\Psi^u_f(K)$ is called the *monodromy weight filtration*. An important theorem of O. Gabber (see [11], §5) characterizes this filtration in the case of l-adic perverse sheaves.

5.7. **Two descriptions of the category of perverse sheaves.** In this section we discuss two descriptions of the category of perverse sheaves on an algebraic variety. Although not strictly necessary for what follows, they play an important role in the theory and applications of perverse sheaves. The question is roughly as follows: suppose X is an algebraic variety, $Y \subseteq X$ a subvariety, and we are given a perverse sheaf K on $X \setminus Y$. How much information is needed to describe the perverse sheaves $\widetilde{K}$ on X whose restriction to $X \setminus Y$ is isomorphic to K? We describe the approach developed by R. MacPherson and K. Vilonen [135] and the approach of A. Beilinson and J.L. Verdier [8, 172].

5.7.1. *The approach of MacPherson-Vilonen.* We report on only a part of the description of the category of perverse sheaves developed in [135], i.e., the most elementary and the one which we find particularly illuminating.

Assume that $X = Y \amalg (X\setminus Y)$, where Y is a closed and contractible d-dimensional stratum of a stratification Σ of X. We have $\mathcal{P}^\Sigma_X$, i.e., the category of perverse sheaves on X which are constructible with respect to Σ. Denote by $Y \xrightarrow{i} X \xleftarrow{j} X \setminus Y$ the corresponding imbeddings.

For $K \in \mathcal{P}^\Sigma_X$, the attaching triangle $i_!i^!K \longrightarrow K \longrightarrow j_*j^*K \xrightarrow{[1]}$, and the support and co-support conditions for a perverse sheaf give the following exact sequence of local systems on Y:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{H}^{-d-1}(i^*K) \arrow[r] & \mathcal{H}^{-d-1}(i^*j_*j^*K) \arrow[r] & \mathcal{H}^{-d}(i^!K) \arrow[d] \\ 0 & \mathcal{H}^{-d+1}(i^!K) \arrow[l] & \mathcal{H}^{-d}(i^*j_*j^*K) \arrow[l] & \mathcal{H}^{-d}(i^*K). \arrow[l] \end{tikzcd} \tag{41}$$

Note that the (trivial) local systems $\mathcal{H}^{-d-1}(i^*j_*j^*K), \mathcal{H}^{-d}(i^*j_*j^*K)$ are determined by the restriction of K to $X \setminus Y$.

A first approximation to the category of perverse sheaves is given as follows:

Definition 5.7.1. Let $\mathcal{P}'_X$ be the following category:
• an object is a perverse sheaf K on $X \setminus Y$, constructible with respect to $\Sigma_{|X-Y}$, and an exact sequence

$$\mathcal{H}^{-d-1}(i^*j_*K) \longrightarrow V_1 \longrightarrow V_2 \longrightarrow \mathcal{H}^{-d}(i^*j_*K)$$

of local systems on Y;

• a morphism $(K,\dots)$ $(L,\dots)$ is a morphism of perverse sheaves $\phi: K \to L$ together with morphisms of exact sequences:

$$\begin{tikzcd}
\mathcal{H}^{-d-1}(i^*j_*K) \arrow[r] \arrow[d, "\phi"] & V_1 \arrow[r] \arrow[d] & V_2 \arrow[r] \arrow[d] & \mathcal{H}^{-d}(i^*j_*K)) \arrow[d, "\phi"] \\
\mathcal{H}^{-d-1}(i^*j_*L) \arrow[r] & W_1 \arrow[r] & W_2 \arrow[r] & \mathcal{H}^{-d}(i^*j_*L)).
\end{tikzcd}$$

Theorem 5.7.2. *The functor $\mathcal{P}^{\Sigma}_X \to \mathcal{P}'_X$, sending a perverse sheaf $\widetilde{K}$ on X to its restriction to $X \setminus Y$ and to the exact sequence*

$$\mathcal{H}^{-d-1}(i^*j_*j^*\widetilde{K}) \longrightarrow \mathcal{H}^{-d}(i^!\widetilde{K}) \longrightarrow \mathcal{H}^{-d}(i^*\widetilde{K}) \longrightarrow \mathcal{H}^{-d}(i^*j_*j^*\widetilde{K})$$

is a bijection on isomorphism classes of objects.

To give an idea why the theorem is true, we note that for any object Q in $\mathcal{P}_X$, we have the distinguished triangle

$$i_!i^!Q \longrightarrow Q \longrightarrow j_*j^*Q \xrightarrow{[1]},$$

and Q is identified by the extension map $e \in \mathrm{Hom}(j_*j^*Q, i_!i^!Q[1])$. We have $i_! = i_*$; hence

$$\mathrm{Hom}(j_*j^*Q, i_!i^!Q[1]) = \mathrm{Hom}(i^*j_*j^*Q, i^!Q[1]) = \bigoplus_l \mathrm{Hom}(\mathcal{H}^l(i^*j_*j^*Q), \mathcal{H}^{l+1}(i^!Q)).$$

The last equality is due to the fact that the derived category of complexes with constant cohomology sheaves on a contractible space is semisimple ($K \simeq \bigoplus H^i(K)[-i]$, for every K). By the support condition,

$$\mathcal{H}^l(i^*j_*j^*Q) \simeq \mathcal{H}^{l+1}(i^!Q) \text{ for } l > -d.$$

By the co-support condition,

$$\mathcal{H}^l(i^!Q) = 0 \text{ for } l < -d.$$

There are the two maps

$$\mathcal{H}^{-d}(i^*j_*j^*Q) \longrightarrow \mathcal{H}^{-d+1}(i^!Q), \qquad \mathcal{H}^{-d-1}(i^*j_*j^*Q) \longrightarrow \mathcal{H}^{-d}(i^!Q)$$

which are not determined a priori by the restriction of Q to $X \setminus Y$. They appear in the exact sequence (41) and contain the information about how to glue j^*Q to $i^!Q$. The datum of this exact sequence makes it possible to reconstruct $Q \in \mathcal{P}_X$ satisfying the support and co-support conditions.

Unfortunately the functor is not as precise on maps, as we will see. There are nonzero maps between perverse sheaves which induce the zero map in $\mathcal{P}'_X$; i.e., the corresponding functor is not faithful. However, it is interesting to see a few examples of applications of this result.

Example 5.7.3. Let $X = \mathbb{C}$, $Y = \{o\}$ with strata $X \setminus Y = \mathbb{C}^*$ and Y. A perverse sheaf on $\mathbb{C}^*$ is then of the form $L[1]$ for L a local system. Let L denote the stalk of L at some base point, and $T: L \to L$ the monodromy. An explicit computation shows that

$$i^*j_*L[1] \simeq \mathrm{Ker}\,(T-I)[1] \oplus \mathrm{Coker}\,(T-I),$$

where $\mathrm{Ker}(T-I)$ and $\mathrm{Coker}(T-I)$ are interpreted as sheaves on Y. Hence a perverse sheaf is identified up to isomorphism by L and by an exact sequence of vector spaces:

$$\mathrm{Ker}\,(T-I) \longrightarrow V_1 \to V_2 \longrightarrow \mathrm{Coker}\,(T-I).$$

A sheaf of the form i_*V is represented by $L=0$ and by the sequence

$$0 \longrightarrow V \xrightarrow{\simeq} V \longrightarrow 0.$$

Since j is an affine imbedding, j_* and $j_!$ are t-exact; i.e., $j_*L[1]$ and $j_!L[1]$ are perverse.

The perverse sheaf $j_*L[1]$ is represented by

$$\mathrm{Ker}\,(T-I) \longrightarrow 0 \longrightarrow \mathrm{Coker}\,(T-I) \xrightarrow{Id} \mathrm{Coker}(T-I),$$

which expresses the fact that $i^!j_*L[1]=0$.

Similarly $j_!L[1]$, which satisfies $i^*j_!L[1]=0$, is represented by

$$\mathrm{Ker}\,(T-I) \xrightarrow{Id} \mathrm{Ker}\,(T-I) \longrightarrow 0 \longrightarrow \mathrm{Coker}\,(T-I).$$

The intermediate extension $j_{!*}L[1]$ is represented by

$$\mathrm{Ker}\,(T-I) \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathrm{Coker}\,(T-I),$$

since, by its very definition,

$$\mathcal{H}^0(i^*j_{!*}L[1]) = \mathcal{H}^0(i^!j_{!*}L[1]) = 0.$$

Let us note another natural exact sequence given by

$$\mathrm{Ker}\,(T-I) \longrightarrow L \xrightarrow{T-I} L \longrightarrow \mathrm{Coker}\,(T-I)$$

which corresponds to Beilinson's maximal extension $\Xi(L)$, which will be described in the next section. From these presentations one sees easily the natural maps

$$j_!L[1] \longrightarrow j_{!*}L[1] \longrightarrow j_*L[1] \qquad \text{and} \qquad j_!L[1] \longrightarrow \Xi(L[1]) \longrightarrow j_*L[1].$$

Remark 5.7.4. If T has no eigenvalue equal to one, then the sequence has the form $0 \to V \to V \to 0$. This corresponds to the fact that a perverse sheaf which restricts to such a local system on $\mathbb{C}\setminus\{o\}$ is necessarily of the form $j_!L[1] \oplus i_*V$. Note also that $j_!L[1] = j_*L[1] = j_{!*}L[1]$.

Remark 5.7.5. One can use Theorem 5.7.2 to deduce the following special case of a splitting criterion used in our proof of the decomposition theorem [51]: Let $d=\dim Y$. A perverse sheaf $K \in \mathcal{P}_X$ splits as $K \simeq j_{!*}j^*K \oplus \mathcal{H}^{-d}(K)[d]$ if and only if the map $\mathcal{H}^{-d}(i^!K) \to \mathcal{H}^{-d}(i^*K)$ is an isomorphism.

In fact, if this condition is verified, then the maps $\mathcal{H}^{-d-1}(i^*j_*j^*K) \to \mathcal{H}^{-d}(i^!K)$ and $\mathcal{H}^{-d}(i^*K) \to \mathcal{H}^{-d}(i^*j_*j^*K)$ in (41) vanish, and the exact sequence corresponding to K is of the form

$$\begin{array}{ccc}
\mathcal{H}^{-d-1}(i^*j_*K) \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathcal{H}^{-d}(i^*j_*K)) & & j_{!*}j^*K \\
 & & \\
\oplus & & \\
 & & \\
W \longrightarrow W & & \mathcal{H}^{-d}(K)[d].
\end{array}$$

The following example shows that the functor $\mathcal{P}_X^\Sigma \to \mathcal{P}'_X$ is not faithful. Consider the perverse sheaf $j_*\mathbb{Q}_{\mathbb{C}^*}[1]$. It has a nonsplit filtration by perverse sheaves

$$0 \longrightarrow \mathbb{Q}_{\mathbb{C}}[1] \longrightarrow j_*\mathbb{Q}_{\mathbb{C}^*}[1] \xrightarrow{\alpha} i_*\mathbb{Q}_0 \longrightarrow 0.$$

Dually, the perverse sheaf $j_!\mathbb{Q}[1]$ has a nonsplit filtration

$$0 \longrightarrow i_*\mathbb{Q}_0 \xrightarrow{\beta} j_!\mathbb{Q}_{\mathbb{C}^*}[1] \longrightarrow \mathbb{Q}_{\mathbb{C}}[1] \longrightarrow 0.$$

The composition $\beta\alpha : j_*\mathbb{Q}_{\mathbb{C}^*}[1] \to j_!\mathbb{Q}_{\mathbb{C}^*}[1]$ is not zero, being the composition of the epimorphism α with the monomorphism β; however, it is zero on $\mathbb{C}^*$, and the map between the associated exact sequences is zero, since $i^!j_*\mathbb{Q}_{\mathbb{C}^*}[1] = 0$ and $i^*j_!\mathbb{Q}_{\mathbb{C}^*}[1] = 0$.

In the paper [135], MacPherson and Vilonen give a refinement of the construction which describes completely the category of perverse sheaves, both in the topological and complex analytic situation. For an application to representation theory, see [143].

5.7.2. *The approach of Beilinson and Verdier.* We turn to Beilinson's approach [7], i.e., the one used by Saito in his theory of mixed Hodge modules. Beilinson's approach is based on the nearby and vanishing cycle functors Ψ_f and Φ_f introduced in §5.5. In [173], Verdier obtained similar results using the specialization to the normal cone functor $\mathrm{Sp}_{Y,X}$, Remark 5.5.4, which is not discussed here.

The assumption is that we have an algebraic map $f : X \to \mathbb{C}$ and $X_0 = f^{-1}(0)$ as in §5.5. Let K be a perverse sheaf on $X \setminus X_0$. Beilinson defines an interesting extension of K to X which he calls the maximal extension and denotes by $\Xi(K)$. It is a perverse sheaf, restricting to K on $X \setminus X_0$, which can be constructed as follows: consider the unipotent nearby and vanishing cycle functors Ψ_f^u and Φ_f^u (see §5.6) and the distinguished triangle

$$i^*j_*K \xrightarrow{[1]} \Psi_f^u(K) \xrightarrow{T-I} \Psi_f^u(K) \longrightarrow \cdot$$

The natural map $i^*j_*K \to \Psi_f^u(K)[1]$ defines, by adjunction, an element of

$$\mathrm{Hom}^1_{\mathcal{D}_{X_0}}(i^*j_*K, \Psi_f^u(K)) = \mathrm{Hom}^1_{\mathcal{D}_X}(j_*K, i_*\Psi_f^u(K)),$$

which, in turn, defines an object $\Xi(K)$ fitting in the distinguished triangle

$$i_*\Psi_f^u(K) \longrightarrow \Xi(K) \longrightarrow j_*K \longrightarrow i_*\Psi_f^u(K)[1]. \tag{42}$$

Since j is an affine morphism, it follows that j_*K is perverse. The long exact sequence of perverse cohomology implies that $\Xi(K)$ is perverse as well.

In [7], Beilinson gives a different construction of $\Xi(K)$ (and also of $\Psi_f^u(K)$ and $\Phi_f^u(K)$) which implies automatically that Ξ is a functor and that it commutes with Verdier duality.

There exists an exact sequence of perverse sheaves

$$0 \longrightarrow i_*\Psi_f^u(K) \xrightarrow{\beta_+} \Xi(K) \xrightarrow{\alpha_+} j_*K \longrightarrow 0$$

and, applying Verdier duality and the canonical isomorphisms $\Xi \circ D \simeq D \circ \Xi$ and $\Psi_f^u \circ D \simeq D \circ \Psi_f^u$,

$$0 \longrightarrow j_!K \xrightarrow{\alpha_-} \Xi(K) \xrightarrow{\beta_-} i_*\Psi_f^u(K) \longrightarrow 0.$$

The composition $\alpha_+\alpha_- : j_!K \to j_*K$ is the natural map, while $\beta_-\beta_+ : i_*\Psi^u_f(K) \to i_*\Psi^u_f(K)$ is $T-I$. We may now state Beilinson's results.

Definition 5.7.6. Let $Gl(X,Y)$ be the category whose objects are quadruples (K_U, V, u, v), where K_U is a perverse sheaf on $U := X \setminus X_0$, V is a perverse sheaf on Y, $u : \Psi^u_f(K) \to V$, and $v : V \to \Psi^u_f(K)$ are morphisms such that $vu = T - I$.

Theorem 5.7.7. *The functor $\gamma : \mathcal{P}_X \to Gl(X,Y)$ which associates to a perverse sheaf K on X the quadruple $(j^*K, \Phi^u_f(K), can, var)$ is an equivalence of categories. Its inverse is the functor $G : Gl(X,Y) \to \mathcal{P}_X$ associating to (K_U, V, u, v) the cohomology of the complex*

$$\Psi^u_f(K_U) \xrightarrow{(\beta_+,u)} \Xi(K_U) \oplus V \xrightarrow{(\beta_-,v)} \Psi^u_f(K_U).$$

Example 5.7.8. Given a perverse sheaf K_U on $U = X \setminus X_0$, we determine

$$\gamma(j_!K_U) \longrightarrow \gamma(j_{!*}K_U) \longrightarrow \gamma(j_*K_U).$$

We make use of the distinguished triangles (39) and (40) discussed in §5.5 and restricted to the unipotent parts Ψ^u_f and Φ^u_f. Since $i^*j_!K_U = 0$, the map *can*: $\Psi^u_f(j_!K_U) \to \Phi^u_f(j_!K_U)$ is an isomorphism. Hence

$$\gamma(j_!K_U) = \Psi^u_f(K_U) \xrightarrow{\mathrm{Id}} \Psi^u_f(K_U) \xrightarrow{T-I} \Psi^u_f(K_U).$$

Similarly, since $i^!j_*K_U = 0$, the map $var : \Phi^u_f(j_*K_U) \longrightarrow \Psi^u_f(j_*K_U)$ is an isomorphism, and

$$\gamma(j_*K_U) = \Psi^u_f(K_U) \xrightarrow{T-I} \Psi^u_f(K_U) \xrightarrow{id} \Psi^u_f(K_U).$$

The canonical map $j_!K_U \to j_*K_U$ is represented by the following diagram, in which we do not indicate the identity maps:

$$\begin{tikzcd}
\gamma(j_!K_U) \arrow[d] & \Psi^u_f(K_U) \arrow[r] \arrow[d] & \Psi^u_f(K_U) \arrow[r, "T-I"] \arrow[d, "T-I"] & \Psi^u_f(K_U) \arrow[d] \\
\gamma(j_*K_U) & \Psi^u_f(K_U) \arrow[r, "T-I"] & \Psi^u_f(K_U) \arrow[r] & \Psi^u_f(K_U).
\end{tikzcd} \tag{43}$$

The intermediate extension $j_{!*}K_U$ corresponds to $j_{!*}K_U := \mathrm{Im}\{\, j_!K_U \to j_*K_U \,\}$; hence

$$\gamma(j_{!*}K_U) = \qquad \Psi^u_f(K_U) \xrightarrow{T-I} \mathrm{Im}(T-I) \hookrightarrow \Psi^u_f(K_U),$$

where the second map is the canonical inclusion. We can complete the diagram (43) as follows:

$$\begin{tikzcd}
\gamma(j_!K_U) \arrow[d] & \Psi^u_f(K_U) \arrow[r] \arrow[d] & \Psi^u_f(K_U) \arrow[r, "T-I"] \arrow[d, "T-I"] & \Psi^u_f(K_U) \arrow[d] \\
\gamma(j_{!*}K_U) \arrow[d] & \Psi^u_f(K_U) \arrow[r, "T-I"] \arrow[d] & \mathrm{Im}(T-I) \arrow[r, hook] \arrow[d] & \Psi^u_f(K_U) \arrow[d] \\
\gamma(j_*K_U) & \Psi^u_f(K_U) \arrow[r, "T-I"] & \Psi^u_f(K_U) \arrow[r] & \Psi^u_f(K_U).
\end{tikzcd} \tag{44}$$

The maximal extension $\Xi(K_U)$ is represented by the factorization

$$\Psi_f^u(K_U) \xrightarrow{(I,T-I)} \Psi_f^u(K_U) \oplus \Psi_f^u(K_U) \xrightarrow{p_2} \Psi_f^u(K_U),$$

where $p_2((a_1,a_2)) = a_2$ is the projection on the second factor. Finally we note that if L is a perverse sheaf on X_0, then, since $\Psi_f(i_*L) = 0$,

$$\gamma(i_*L) = 0 \longrightarrow L \longrightarrow 0.$$

Remark 5.7.9. From the examples of $\gamma(j_{!*}K_U)$ and $\gamma(i_*L)$ discussed in Example 5.7.8, one can derive the following criterion (Lemme 5.1.4 in [156]) for a perverse sheaf K on X to split as $K \simeq j_{!*}j^*K \oplus i_*L$. Let X be an algebraic variety and X_0 be a principal divisor; let $i : X_0 \to X \longleftarrow X \setminus X_0 : j$ be the corresponding closed and open imbeddings. A perverse sheaf K on X is of the form $K \simeq j_{!*}j^*K \oplus i_*L$ if and only if

$$\Phi_f^u(K) = \{\mathrm{Im} : (\Psi_f^u(K) \stackrel{\mathrm{can}}{\to} \Phi_f^u(K))\} \oplus \{\mathrm{Ker} : (\Phi_f^u(K) \stackrel{\mathrm{var}}{\to} \Psi_f^u(K))\}.$$

This criterion is used in [156] to establish the semisimplicity of certain perverse sheaves.

5.8. **A formulary for the constructible derived category.** Throughout this section, $f : X \to Y$, $g : Y' \to Y$ and $h : Y \to Z$ are maps of varieties, $C \in \mathcal{D}_X$ is a constructible complex on X and $K, K', K_i \in \mathcal{D}_Y$ are constructible complexes on Y. An equality sign actually stands for the existence of a suitably canonical isomorphism. Since we use field coefficients, the tensor product is exact and it coincides with the associated left derived functor. Perversity means middle perversity on complex varieties. All operations preserve stratifications of varieties and of maps. We use the simplified notation $f_* := Rf_*$, $f_! := Rf_!$. Some standard references are [116, 87, 19, 108, 82, 9, 84].

Cohomology via map to a point or space.

$$H(X,C) = H(pt, f_*C), \qquad H_c(X,C) = H(pt, f_!C);$$
$$H(X,C) = H(Y, f_*C), \qquad H_c(X,C) = H_c(Y, f_!C).$$

Translation functors. Let $T := f^*, f_*, f_!$ or $f^!$:

$T \circ [j] = [j] \circ T.$

${}^{\mathfrak{p}}\tau_{\leq i} \circ [j] = [j] \circ {}^{\mathfrak{p}}\tau_{\leq i+j}, \qquad {}^{\mathfrak{p}}\tau_{\geq i} \circ [j] = [j] \circ {}^{\mathfrak{p}}\tau_{\geq i+j}; \qquad$ same for τ.

$\mathcal{H}^i \circ [j] = \mathcal{H}^{i+j}, \qquad {}^{\mathfrak{p}}\mathcal{H}^i \circ [j] = {}^{\mathfrak{p}}\mathcal{H}^{i+j}.$

$\mathrm{RHom}(K,K')[j] = \mathrm{RHom}(K,K'[j]) = \mathrm{RHom}(K[-j],K').$

$R\mathcal{H}om(K,K')[j] = R\mathcal{H}om(K,K'[j]) = R\mathcal{H}om(K[-j],K').$

$(K \otimes K')[j] = K \otimes K'[j] = K[j] \otimes K'.$

Morphism in $\mathcal{D}_Y$.

$$\begin{aligned}\mathrm{Ext}^i_{\mathcal{D}_Y}(K,K') = \mathrm{Hom}_{\mathcal{D}_Y}(K,K'[i]) &= H^0(\mathrm{RHom}(K,K'[i])) \\ &= H^0(Y, R\mathcal{H}om(K,K'[i])).\end{aligned}$$

If $K \in {}^{\mathfrak{p}}\mathcal{D}_Y^{\leq i}$ and $K' \in {}^{\mathfrak{p}}\mathcal{D}_Y^{\geq i}$, then (same for the standard t-structure)

$$\mathrm{Hom}_{\mathcal{D}_Y}(K,K') = \mathrm{Hom}_{\mathcal{P}_Y}({}^{\mathfrak{p}}\mathcal{H}^i(K), {}^{\mathfrak{p}}\mathcal{H}^i(K')).$$

For sheaves, $\mathrm{Ext}^{<0}(F,G)=0$ and, $\mathrm{Ext}^{i>0}(F,G)$ is the group of Yoneda i-extensions of G by F. The group $\mathrm{Ext}^1(F,G)$ is the set of equivalence classes of short exact sequences $0 \to F \to ? \to G \to 0$ with the Baer sum operation. For complexes, $\mathrm{Ext}^1(K,K')$ classifies distinguished triangles $K \to ? \to K' \to K[1]$.

Adjunction.

$$\mathrm{RHom}(f^*K,C)=\mathrm{RHom}(K,f_*C), \qquad \mathrm{RHom}(f_!C,K)=\mathrm{RHom}(C,f^!K),$$
$$\mathrm{RHom}(K_1\otimes K_2,K_3)=\mathrm{RHom}(K_1,R\mathcal{H}om(K_2,K_3));$$
$$f_*R\mathcal{H}om(f^*K,C)=R\mathcal{H}om(K,f_*C), \qquad R\mathcal{H}om(f_!C,K)=f_*R\mathcal{H}om(C,f^!K),$$
$$R\mathcal{H}om(K_1\otimes K_2,K_3)=R\mathcal{H}om(K_1,R\mathcal{H}om(K_2,K_3)).$$

If all $\mathcal{H}^j(K_3)$ are locally constant, then

$$R\mathcal{H}om(K_1,K_2\otimes K_3)=R\mathcal{H}om(K_1,K_2)\otimes K_3.$$

Transitivity.

$$(hf)_*=h_*f_*, \quad (hf)_!=h_!f_!, \quad (hf)^*=f^*h^*, \quad (hf)^!=f^!h^!,$$
$$f^*(K\otimes K')=f^*K\otimes f^*K', \qquad f^!R\mathcal{H}om(K,K')=R\mathcal{H}om(f^*K,f^!K').$$

Change of coefficients.

$$K\otimes f_!C\simeq f_!(f^*K\otimes C).$$

Duality exchanges.

$$DK:=K^\vee:=R\mathcal{H}om(K,\omega_Y), \qquad \omega_Y:=\gamma^!\mathbb{Q}_{pt}, \quad \gamma: Y\to pt.$$
$$D: {}^{\mathfrak{p}}\mathcal{D}^{\leq 0}_Y \longrightarrow {}^{\mathfrak{p}}\mathcal{D}^{\geq 0}_Y, \qquad D: {}^{\mathfrak{p}}\mathcal{D}^{\geq 0}_Y \longrightarrow {}^{\mathfrak{p}}\mathcal{D}^{\leq 0}_Y, \qquad D:\mathcal{P}_Y\simeq \mathcal{P}_Y^{opp}.$$

If $F:\mathcal{D}_X\to\mathcal{D}_Y$ is left (right, resp.) t-exact, then $D\circ F\circ D$ is right (left, resp.) t-exact. A similar result holds for $G:\mathcal{D}_Y\to\mathcal{D}_X$.

$$\omega_Y=\mathbb{Q}_Y^\vee;$$
$$D\circ[j]=[-j]\circ D;$$
$$D_Y\circ f_*=f_!\circ D_X, \qquad D_X\circ f^*=f^!\circ D_Y;$$
$$D\circ {}^{\mathfrak{p}}\tau_{\leq j}={}^{\mathfrak{p}}\tau_{\geq -j}\circ D, \qquad D\circ {}^{\mathfrak{p}}\tau_{\geq j}={}^{\mathfrak{p}}\tau_{\leq -j}\circ D, \qquad {}^{\mathfrak{p}}\mathcal{H}^j\circ D=D\circ {}^{\mathfrak{p}}\mathcal{H}^{-i};$$
$$D(K\otimes K')=R\mathcal{H}om(K,DK');$$
$$D^2=Id \quad \text{(biduality)}.$$

Poincaré-Verdier duality.

$$H^j(Y,DK)\simeq H_c^{-j}(Y,K)^\vee.$$

If Y is smooth of pure complex dimension n and is canonically oriented, then

$$\omega_Y=\mathbb{Q}_Y[-2n].$$

Support conditions for perverse sheaves.

Support conditions: $K\in {}^{\mathfrak{p}}\mathcal{D}^{\leq 0}_Y$ iff $\dim \mathrm{Supp}\,\mathcal{H}^i(K)\leq -i$, for every i.
Co-support conditions: $K\in {}^{\mathfrak{p}}\mathcal{D}^{\geq 0}_Y$ iff $\dim \mathrm{Supp}\,\mathcal{H}^i(DK)\leq -i$, for every i.
A perverse sheaf is a complex subject to the support and co-support conditions.

Base change. Consider the Cartesian square, where the ambiguity of the notation does not generate ambiguous statements:

$$\begin{array}{ccc} X' & \xrightarrow{g} & X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y. \end{array}$$

Base change isomorphisms:

$$g^! f_* = f_* g^!, \qquad f_! g^* = g^* f_!.$$

For the immersion of a point $g : y \to Y$,

$$H^l_c(f^{-1}(y), C) \,=\, (R^l f_! C)_y\,; \qquad\qquad H^l(f^{-1}(y), C) \,=\, (R^l f_* C)_y \quad (f \text{ proper}).$$

Base change maps:

$$g^* f_* \longrightarrow f_* g^*, \qquad f_! g^! \simeq g^! f_!.$$

Proper (smooth, resp.) base change: if f is proper (g is smooth, resp.), then the base change maps are isomorphisms.

There exist natural maps

$$g_! f_* \longrightarrow f_* g_!, \qquad f_! g_* \longrightarrow g_* f_!.$$

Intermediate extension functor. For f a locally closed embedding,

$$f_{!*} : \mathcal{P}_X \longrightarrow \mathcal{P}_Y, \quad P \longmapsto \operatorname{Im}\{\, {}^{\mathfrak{p}}\!\mathcal{H}^0(f_! P) \longrightarrow {}^{\mathfrak{p}}\!\mathcal{H}^0(f_* P)\}.$$

For an open immersion, the intermediate extension is characterized as the extension with no subobjects and no quotients supported on the boundary (however, it may have such subquotients).

Intersection cohomology complexes. Let L be a local system on a nonsingular Zariski dense open subset $j : U \to Y$ of the irreducible n-dimensional Y. Then

$$IC_Y(L) := j_{!*} L[n] \,\in\, \mathcal{P}_Y.$$

If the smallest dimension of a stratum is d, then

$$\mathcal{H}^l(IC_Y(L)) = 0, \quad \forall j \neq [-n, -d-1];$$

note that for a general perverse sheaf, the analogous range is $[-n, -d]$.
As to duality:

$$D(IC_Y(L)) = IC_Y(L^\vee).$$

The category $\mathcal{P}_Y$ is Artinian and Noetherian. The simple objects are the intersection cohomology complexes of simple local systems on irreducible subvarieties.

Nearby and vanishing cycles. With a regular function $f : Y \to \mathbb{C}$ are associated the two functors $\Psi_f, \Phi_f : \mathcal{D}_Y \to \mathcal{D}_{Y_0}$, where $Y_0 = f^{-1}(0)$. If $Y \setminus Y_0 \xrightarrow{j} Y \xleftarrow{i} Y_0$, there are distinguished triangles:

$$i^* K \longrightarrow \Psi_f(K) \xrightarrow{\text{can}} \Phi_f(K)[1] \xrightarrow{[1]}, \tag{45}$$

$$i^! K \longrightarrow \Phi_f(K) \xrightarrow{\text{var}} \Psi_f(K)[-1] \xrightarrow{[1]} .$$

The functors Ψ_f, Φ_f are endowed with the monodromy automorphism T and

$$\text{can} \circ \text{var} = T - I : \Phi_f(K) \to \Phi_f(K), \qquad \text{var} \circ \text{can} = T - I : \Psi_f(K) \to \Psi_f(K).$$

Consider the distinguished triangle

$$i^* j_* j^* K \longrightarrow \Psi_f(K) \xrightarrow{T-I} \Psi_f(K) \xrightarrow{[1]} .$$

Up to a shift, the functors Ψ_f, Φ_f commute with duality and are t-exact:

$$\Psi_f \circ D = D \circ \Psi_f \circ [2], \quad \Phi_f \circ D = D \circ \Phi_f \circ [2], \qquad \Psi_f[-1], \Phi_f[-1] : \mathcal{P}_Y \longrightarrow \mathcal{P}_{Y_0}.$$

For $K \in \mathcal{P}_{Y \setminus Y_0}$, the long exact sequence for the distinguished triangle above gives:

$$\begin{aligned} {}^{\mathfrak{p}}\mathcal{H}^{-1}(i^* j_* K) &= \operatorname{Ker}\{\, \Psi_f(K)[-1] \stackrel{T-I}{\longrightarrow} \Psi_f(K)[-1] \,\}, \\ {}^{\mathfrak{p}}\mathcal{H}^{0}(i^* j_* K) &= \operatorname{Coker}\{\, \Psi_f(K)[-1] \stackrel{T-I}{\longrightarrow} \Psi_f(K)[-1] \,\}, \end{aligned}$$

$j_* K$ and $j_! K \in \mathcal{P}_Y$, and comparing the above equalities with the distinguished triangle:

$$i^* j_* K \xrightarrow{[1]} j_! K \longrightarrow j_* K \longrightarrow$$

yields

$$\begin{aligned} \operatorname{Ker}\{\, j_! K \to j_{!*} K \,\} &\simeq \operatorname{Ker}\{\, \Psi_f(K)[-1] \stackrel{T-I}{\longrightarrow} \Psi_f(K)[-1] \,\}, \\ \operatorname{Coker}\{\, j_{!*} K \to j_* K \,\} &\simeq \operatorname{Coker}\{\, \Psi_f(K)[-1] \stackrel{T-I}{\longrightarrow} \Psi_f(K)[-1] \,\}. \end{aligned}$$

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About the authors

Mark Andrea A. de Cataldo is a professor at SUNY Stony Brook. He was an AMS Centennial Fellow and he has been a member of the Institute For Advanced Study in Princeton.

Luca Migliorini is professor of geometry at the University of Bologna, Italy.

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Department of Mathematics, Stony Brook University, Stony Brook, New York 11794
E-mail address: mde@math.sunysb.edu

Dipartimento di Matematica, Università di Bologna, Piazza di Porta S. Donato 5, 40126 Bologna, Italy
E-mail address: migliori@dm.unibo.it